Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Have a quick look back at the first lecture to remind yourself how the Riemann integral was constructed. You will now see that the Lebesgue integral is constructed rather differently.

Definition: Let $(X, \mathcal{X}, \mu)$ be a measure space. Let

$$s(x) = \sum_{j=1}^{k} a_j \, \chi_{E_j}(x)$$

be a simple function. We say that the simple function s is in the *standard representation* if the a_j are distinct and the E_j are pairwise disjoint. Without saying so explicitly, we will usually assume that our simple functions are in standard representation.

Now we define the integral of s with respect to the measure μ to be

$$\int s \, d\mu = \sum_{j=1}^{k} a_j \, \mu(E_j) \, . \qquad (1)$$

As noted previously, we adhere to the custom that $0 \cdot +\infty = 0$. So, for instance, the integral of the identically 0 function equals 0. It is certainly possible for the value of the integral in formula (1) to take the value $+\infty$—for instance if $a_1 = 1$ and $\mu(E_1) = +\infty$.

Example: Let μ be Lebesgue measure on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 2 & \text{if} \quad -1 < x < 1 \\ 3 & \text{if} \quad 3 < x < 7 \\ -1 & \text{if} \quad -4 \le x < -3 \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\int f \, d\mu(x) = 2 \cdot (1 - (-1)) + 3 \cdot (7 - 3) + (-1) \cdot ((-3) - (-4)) = 15 .$$

Now we develop some elementary properties of the integral.

Lemma: *Let $(X, \mathcal{X}, \mu)$ be a measure space. If φ, ψ are simple, nonnegative functions and if $c \geq 0$, then*

$$\int c\varphi \, d\mu = c \int \varphi \, d\mu \tag{1}$$

and

$$\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu \,. \tag{2}$$

Furthermore, if λ is defined for $E \in \mathcal{X}$ by

$$\lambda(E) = \int \varphi \cdot \chi_E \, d\mu \,, \tag{3}$$

then λ is a measure on $\mathcal{X}$.

Proof: If $c = 0$ then equation (1) becomes trivial. So suppose $c > 0$. Then $c\varphi$ is a nonnegative, simple function. If

$$\varphi(x) = \sum_{j=1}^{k} a_j \cdot \chi_{E_j}(x)\,,$$

then

$$c\varphi(x) = \sum_{j=1}^{k} ca_j \cdot \chi_{E_j}(x)\,.$$

Thus

$$\int c\varphi\,d\mu = \sum_{j=1}^{k} ca_j \cdot \mu(E_j) = c\sum_{j=1}^{k} a_j \cdot \mu(E_j) = c\int \varphi\,d\mu\,.$$

That establishes (1).

Suppose now that

$$\varphi = \sum_{j=1}^{k} a_j \cdot \chi_{E_j} \quad \text{and} \quad \psi = \sum_{\ell=1}^{m} b_\ell \cdot \chi_{F_\ell}$$

with the E_j pairwise disjoint and the F_ℓ pairwise disjoint. Then $\varphi + \psi$ has the representation

$$\begin{aligned} \varphi + \psi = & \sum_{j=1}^{k} \sum_{\ell=1}^{m} (a_j + b_\ell) \cdot \chi_{E_j \cap F_\ell} \\ & + \sum_{j=1}^{k} \sum_{\ell=1}^{m} a_j \cdot \chi_{E_j \setminus (F_1 \cup \cdots \cup F_m)} \\ & + \sum_{j=1}^{k} \sum_{\ell=1}^{m} b_\ell \cdot \chi_{F_\ell \setminus (E_1 \cup \cdots \cup E_k)} \,. \end{aligned} \tag{4}$$

This last representation for $\varphi + \psi$ is a bit confusing since different occurrences of $a_j + b_\ell$ may be equal (so that (4) is not necessarily the standard representation of simple function given in the original definition). Let c_p, $p = 1, 2, \ldots, r$, be the distinct numbers in the collection $\{a_j + b_\ell : j = 1, \ldots, k; \ell = 1, \ldots, m\}$. Let H_p be the union of all those sets $E_j \cap F_\ell \neq \emptyset$ so that $a_j + b_\ell = c_p$. Then

$$\mu(H_p) = \sum_{(p)} \mu(E_j \cap F_\ell) .$$

Here the notation (p) means that we sum over choices of j and ℓ so that $a_j + b_\ell = c_p$ and $E_j \cap F_\ell \neq \emptyset$. Of course the standard representation of $\varphi + \psi$ is now given by

$$\varphi + \psi = \sum_{p=1}^{r} c_p \cdot \chi_{H_p} + \sum_{j=1}^{k} a_j \cdot \chi_{E_j \setminus (F_1 \cup F_2 \cup \cdots \cup F_m)} + \sum_{\ell=1}^{m} b_\ell \cdot \chi_{F_\ell \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)} .$$

As a result, we may now calculate that

$$\begin{aligned}
\int (\varphi + \psi)\, d\mu &= \sum_{p=1}^{r} c_p \cdot \mu(H_p) + \sum_{j=1}^{k} a_j \cdot \mu[E_j \setminus (F_1 \cup F_2 \cup \cdots \cup F_m)] \\
&\qquad + \sum_{\ell=1}^{m} b_\ell \cdot \mu[F_\ell \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)] \\
&= \sum_{p=1}^{r} \sum_{(p)} c_p \cdot \mu(E_j \cap F_\ell) \\
&\quad + \sum_{j=1}^{k} a_j \cdot \mu[E_j \setminus (F_1 \cup F_2 \cup \cdots \cup F_m)] \\
&\qquad + \sum_{\ell=1}^{m} b_\ell \cdot \mu[F_\ell \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{r} \sum_{(p)} (a_j + b_\ell) \cdot \mu(E_j \cap F_\ell) \\
&\qquad + \sum_{j=1}^{k} a_j \cdot \mu[E_j \setminus (F_1 \cup F_2 \cup \cdots \cup F_m)] \\
&\qquad + \sum_{\ell=1}^{m} b_\ell \cdot \mu[F_\ell \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)] \\
&= \sum_{j=1}^{k} \sum_{\ell=1}^{m} (a_j + b_\ell) \cdot \mu(E_j \cap F_\ell) + \sum_{j=1}^{k} a_j \cdot \mu[E_j \setminus (F_1 \cup F_2 \cup \cdots \cup F_m)] \\
&\qquad + \sum_{\ell=1}^{m} b_\ell \cdot \mu[F_\ell \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} a_j \cdot \left[\mu(E_j \setminus (F_1 \cup \cdots \cup F_m)) + \sum_{\ell=1}^{m} \mu(E_j \cap F_\ell) \right] \\
&\qquad + \sum_{\ell=1}^{m} b_\ell \cdot \left[\mu(F_\ell \setminus (E_1 \cup \cdots \cup E_k)) + \sum_{j=1}^{k} \mu(E_j \cap F_\ell) \right] \\
&= \sum_{j=1}^{k} a_j \cdot \mu(E_j) + \sum_{\ell=1}^{m} b_\ell \cdot \mu(F_\ell) \\
&= \int \varphi \, d\mu + \int \psi \, d\mu \, .
\end{aligned}
$$

To prove (3), we note that

$$\varphi \chi_E = \sum_{j=1}^{k} a_j \cdot \chi_{E_j \cap E} \, .$$

As a result, we may apply what was proved above to see that

$$\lambda(E) = \int \varphi \chi_E \, d\mu = \sum_{j=1}^{k} a_j \cdot \int \chi_{E_j \cap E} \, d\mu = \sum_{j=1}^{k} a_j \cdot \mu(E_j \cap E) \, .$$

Since the mapping $E \mapsto \mu(E_j \cap E)$ is a measure, we see that we have expressed λ as a linear combination of measures. Thus λ is a measure. $\square$

Now we can define the integral of a nonnegative, measurable function f. The value of this integral could be finite or it could be $+\infty$.

Definition: Let $(X, \mathcal{X}, \mu)$ be a measure space and f a nonnegative, measurable function. Then the *integral of f with respect to μ* is the extended real number

$$\int f \, d\mu \equiv \sup \int s \, d\mu \,,$$

where the supremum is taken over all nonnegative, simple functions s satisfying $0 \leq s(x) \leq f(x)$ for all $x \in \mathbb{R}$.

Definition: Let $(X, \mathcal{X}, \mu)$ be a measure space and f a nonnegative, measurable function. Let $E \in \mathcal{X}$. We define the *integral of f over E with respect to* μ to be

$$\int_E f \, d\mu = \int f \cdot \chi_E \, d\mu \, .$$

This is the *Lebesgue integral of f*.