Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Based on our experience with the Riemann integral, there are certain monotonicity properties that we expect an integral to have:

Proposition: *Let $(X, \mathcal{X}, \mu)$ be a measure space.*

If f and g are nonnegative, measurable functions and $f \leq g$, then

$$\int f \, d\mu \leq \int g \, d\mu \,. \tag{1}$$

If instead f is a nonnegative, measurable function and $E, F \in \mathcal{X}$, and if $E \subseteq F$, then

$$\int_E f \, d\mu \leq \int_F f \, d\mu \, . \tag{2}$$

Proof: If s is a simple, nonnegative function such that $0 \le s \le f$, then it certainly follows that $0 \le s \le g$. Thus (1) holds. Since $f \cdot \chi_E \le f \cdot \chi_F$, the second assertion follows from the first. $\square$

There are three significant theorems about the convergence of the Lebesgue integral. Contrast this situation with that for the Riemann integral—where there is really only one convergence theorem. Now we treat the first of these.

Theorem (The Lebesgue Monotone Convergence Theorem): *Let $(X, \mathcal{X}, \mu)$ be a measure space. Let $\{f_j\}$ be a monotone increasing sequence of nonnegative, measurable functions (i.e., $f_1(x) \le f_2(x) \le \cdots$ for all x) that converge pointwise to a function f. Then*

$$\int f \, d\mu = \lim_{j \to \infty} \int f_j \, d\mu \, . \tag{1}$$

Proof: According to the preceding corollary, the function f is measurable. Since $f_j \leq f_{j+1} \leq f$, we see from the proposition that

$$\int f_j \, d\mu \leq \int f_{j+1} \, d\mu \leq \int f \, d\mu$$

for all $j \in \mathbb{N}$. Thus we have

$$\lim_{j \to \infty} \int f_j \, d\mu \leq \int f \, d\mu \, . \tag{2}$$

This is half of the result. For the opposite inequality, let α be a real number satisfying $0 < \alpha < 1$ and let s be a simple function satisfying $0 \leq s \leq f$. Let

$$A_j = \{x \in X : f_j(x) \geq \alpha s(x)\} \, .$$

Thus $A_j \subseteq X$, $A_j \subseteq A_{j+1}$ for each j, and $X = \cup_j A_j$.

By the proposition,

$$\int_{A_j} \alpha s \, d\mu \leq \int_{A_j} f_j \, d\mu \leq \int_X f_j \, d\mu \, . \tag{3}$$

Since the sequence of sets $\{A_j\}$ is monotone increasing and has union X, we see from the lemma, part **(a)** and (3) that

$$\int s \, d\mu = \lim_{j \to \infty} \int_{A_j} s \, d\mu \, .$$

Thus, taking the limit in (3) with respect to j, we find that

$$\alpha \int s \, d\mu \leq \lim_{j \to \infty} \int f_j \, d\mu \, .$$

Since this inequality holds for all α with $0 < \alpha < 1$, we conclude that

$$\int s\, d\mu \le \lim_{j\to\infty} \int f_j\, d\mu\,.$$

Because s is an arbitrary simple function satisfying $0 \le s \le f$, we find now that

$$\int f\, d\mu = \sup_s \int s\, d\mu \le \lim_{j\to\infty} \int f_j\, d\mu\,.$$

Combining this with (2), we obtain (1). □

Remark: It may be noted that we are not assuming nor asserting that either side of (1) is finite.

Example: Let $(\mathbb{R}, \mathcal{X}, \mu)$ be the usual measure space with $\mathcal{X}$ the Borel sets and μ the Lebesgue measure. Let f be a nonnegative, measurable function such that $\int f \, d\mu$ is finite. Define $f_j = f \cdot \chi_{[0,j]}$. Obviously $\lim_{j \to \infty} f_j(x) = f(x)$ for every x and the convergence is monotone increasing.

Then the lefthand side of (1) is $\int f \, d\mu$. And the righthand side is $\lim_{j \to \infty} \int f_j \, d\mu$. We are guaranteed that the limit in this righthand side equals $\int f \, d\mu$.

Our second convergence result is particularly useful because it applies to sequences of functions that are not monotone.

Theorem (Fatou's Lemma): *Let $(X, \mathcal{X}, \mu)$ be a measure space. Assume that the functions f_j are nonnegative and measurable. Then*

$$\int (\liminf_{j\to\infty} f_j)\, d\mu \le \liminf_{j\to\infty} \int f_j\, d\mu\,. \tag{1}$$

Proof: Let $g_j(x) = \inf\{f_j(x), f_{j+1}(x), \ldots\}$ for each $x \in X$. Then $g_j \le f_k$ whenever $j \le k$. Thus

$$\int g_j \, d\mu \le \int f_k \, d\mu \quad \text{whenever} \quad j \le k \, .$$

Hence

$$\int g_j \, d\mu \le \liminf_{k \to \infty} \int f_k \, d\mu \, .$$

Since the sequence $\{g_j\}$ is increasing and converges to $\liminf_{k\to\infty} f_k$, the monotone convergence theorem tells us that

$$\begin{aligned} \int (\liminf_{k\to\infty} f_k) \, d\mu &= \lim_{j\to\infty} \int g_j \, d\mu \\ &\le \liminf_{k\to\infty} \int f_k \, d\mu \, . \end{aligned}$$

$\square$

Remark: Fatou's lemma is remarkable in part because it says that a liminf is greater than or equal to something else. That is not something that one often sees.

Example: Let $\mathcal{B}$ be the Borel sets in $\mathbb{R}$ and let $d\mu$ be Lebesgue measure. Let $f_j(x) = \chi_{[j,j+1]}$ for $j = 1, 2, \ldots$. Then the lefthand side of (1) is $\int 0 \, d\mu = 0$. And the righthand side of (1) is $\liminf 1 = 1$. Certainly $0 \leq 1$.

Corollary: *If f is a nonnegative, measurable function and if λ is defined on by*

$$\lambda(E) = \int_E f\, d\mu = \int f \cdot \chi_E\, d\mu\,,$$

then λ is a measure.

Proof: Since $f \geq 0$ it is immediate that $\lambda(E) \geq 0$. If $E = \emptyset$, then $f\chi_E$ vanishes everywhere so that $\lambda(\emptyset) = 0$. To see that λ is countably additive, let $\{E_j\}$ be a pairwise disjoint sequence of sets in $\mathcal{X}$ with union E and let f_n be defined to be

$$f_n = \sum_{j=1}^{n} f \cdot \chi_{E_j} \,.$$

Then we see from the additivity of the integral that

$$\int f_n \, d\mu = \sum_{j=1}^{n} \int f \cdot \chi_{E_j} \, d\mu = \sum_{j=1}^{n} \lambda(E_j) \,.$$

Since $\{f_n\}$ is an increasing sequence of nonnegative, measurable functions converging to $f\chi_E$, the monotone convergence theorem tells us that

$$
\begin{aligned}
\lambda(E) &= \int f\chi_E \, d\mu \\
&= \lim_{j\to\infty} \int f_j \, d\mu \\
&= \lim_{j\to\infty} \sum_{\ell=1}^{j} \int f \cdot \chi_{E_\ell} \, d\mu \\
&= \lim_{j\to\infty} \sum_{\ell=1}^{j} \lambda(E_\ell) \\
&= \sum_{\ell=1}^{\infty} \lambda(E_\ell) \, . \qquad \square
\end{aligned}
$$

Corollary: *Let* $(X, \mathcal{X}, \mu)$ *be a measure space. Suppose that* f *is a nonnegative, measurable function. Then* $f(x) = 0$ μ*-almost everywhere if and only if*

$$\int f \, d\mu = 0 \, . \tag{1}$$

Proof: If equation (1) holds, then we let

$$E_j = \left\{ x \in X : f(x) > \frac{1}{j} \right\} .$$

As a result, $f \geq (1/j)\chi_{E_j}$ for each j.
Therefore

$$0 = \int f \, d\mu \geq \int_{E_j} f \, d\mu \geq \int_{E_j} \frac{1}{j} \, d\mu = \frac{1}{j} \cdot \mu(E_j) \geq 0 \, .$$

We conclude that $\mu(E_j) = 0$, hence the set

$$\{x \in X : f(x) > 0\} = \bigcup_{j=1}^{\infty} E_j$$

has measure 0.

For the converse, assume that $f(x) = 0$ almost everywhere. If

$$E = \{x \in X : f(x) > 0\} ,$$

then $\mu(E) = 0$. Set $f_j = j\chi_E$ for $j = 1, 2, \ldots$. Since $f \le \liminf_{j\to\infty} f_j$, Fatou's lemma implies that

$$0 \le \int f \, d\mu \le \int \liminf_{j\to\infty} f_j \, d\mu \le \liminf_{j\to\infty} \int f_j \, d\mu = 0 .$$

Thus $\int f \, d\mu = 0$. $\square$

Corollary: *Let $(X, \mathcal{X}, \mu)$ be a measure space. Let g_j be a sequence of nonnegative, measurable functions. Then*

$$\int \left(\sum_{j=1}^{\infty} g_j \right) \, d\mu = \sum_{j=1}^{\infty} \left(\int g_j \, d\mu \right) .$$

Proof: *Set $f_j = g_1 + g_2 + \cdots + g_j$. Now apply the monotone convergence theorem to the f_j.* $\square$