Math 4121
February 10, 2021 Lecture

Steven G. Krantz

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Figure: This is your instructor.

The Lebesgue Integral

Functions with Finite Integral

In earlier parts of the course we considered the integral of a nonnegative function. In the present chapter we shall treat functions that take both positive and negative values.

Definition: Let $(X, \mathcal{X}, \mu)$ be a measure space. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. We remind the reader that

$$f^+(x) = \max\{f(x), 0\} = \frac{f(x) + |f(x)|}{2}$$

and

$$f^-(x) = \max\{-f(x), 0\} = \frac{|f(x)| - f(x)}{2} .$$

Note that f^+ is the positive part of f and f^- is the negative part of f. Of course we have that $f = f^+ - f^-$.

We call f *integrable* if both f^+ and f^- have finite integral. In that case we set

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu \, .$$

If E is a measurable set then we define

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu = \int f^+ \cdot \chi_E \, d\mu - \int f^- \cdot \chi_E \, d\mu \, .$$

Definition: Let $\mathcal{X}$ be a σ-algebra on $\mathbb{R}$. A *signed measure* on $\mathcal{X}$ is defined to be a function $\lambda : \mathcal{X} \to \mathbb{R}$ so that

(a) $\lambda(\emptyset) = 0$;

(b) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{X}$, then

$$\lambda\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \lambda(E_j)\,.$$

The difference between a signed measure and a measure is that a signed measure can take *any real value* while a measure can only take nonnegative values. For technical reasons, we *do not* allow a signed measure to take the values $\pm\infty$ (while a measure *is* allowed to take the value $+\infty$). We shall usually denote a measure by μ and a signed measure by λ.

Remark: We make it an exercise for the reader to check the following. Suppose that f is as in the definition. Also assume that $f = f_1 - f_2$, with both f_1 and f_2 nonnegative functions having finite integral. Then

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu \, .$$

Lemma: *If f is integrable and $\lambda : \mathcal{X} \to \mathbb{R}$ is defined by*

$$\lambda(E) = \int_E f \, d\mu \, ,$$

then λ is a signed measure.

Proof: Since f^+ and f^- are positive, measurable functions, then an earlier corollary tells us that

$$\lambda^+(E) \equiv \int_E f^+ \, d\mu \quad \text{and} \quad \lambda^-(E) \equiv \int_E f^- \, d\mu$$

are measures on $\mathcal{X}$. They are finite because f is integrable. Since $\lambda = \lambda^+ - \lambda^-$, it follows that λ is a signed measure. $\square$

Theorem: *A measurable function f is integrable if and only if $|f|$ is integrable. In this case,*

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu \, . \tag{1}$$

Proof: By definition, f is integrable if and only if both f^+ and f^- have finite integral. Since

$$|f|^+ = |f| = f^+ + f^-$$

and since

$$|f|^- = 0\,,$$

we see that the proposition and the additivity of the integral imply the asserted inequality.

That is to say,

$$\begin{aligned}\left|\int f\,d\mu\right| &= \left|\int f^+\,d\mu - \int f^-\,d\mu\right| \\ &\le \int f^+\,d\mu + \int f^-\,d\mu \\ &= \int |f|\,d\mu\,.\end{aligned}$$

□

Corollary: *If f is measurable, g is integrable, and $|f| \leq g$, then f is integrable and*

$$\int |f| \, d\mu \leq \int g \, d\mu \, .$$

Proof: *The result is immediate from the proposition.* □

Remark: It follows from our earlier discussions that the integral respects scalar multiplication and addition. We shall say no more about the matter at this time.

We next treat the most important and versatile convergence theorem for integrable functions.