Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

More on Functions with Finite Integral

Theorem [Lebesgue Dominated Convergence Theorem]: *Let $\{f_j\}$ be a sequence of integrable functions which converges almost everywhere to a measurable function f. If there is an integrable function g such that $|f_j| \le g$ for all j, then f is integrable and*

$$\int f \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu \, . \tag{1}$$

Proof: By simply redefining f_j, f to equal 0 on a set of measure 0, we may assume that the convergence takes place on all of X. We may infer from the corollary that f is integrable. Since $g + f_j \geq 0$ for each j, we may apply Fatou's lemma to find that

$$\begin{aligned}
\int g \, d\mu + \int f \, d\mu &= \int (g + f) \, d\mu \\
&= \int \liminf_{j \to \infty} (g + f_j) \, d\mu \\
&\leq \liminf_{j \to \infty} \int (g + f_j) \, d\mu \\
&= \liminf_{j \to \infty} \left(\int g \, d\mu + \int f_j \, d\mu \right) \\
&= \int g \, d\mu + \liminf \int f_j \, d\mu \, .
\end{aligned}$$

It follows that

$$\int f\,d\mu \le \liminf_{j\to\infty} \int f_j\,d\mu\,. \tag{2}$$

Since $g - f_j \ge 0$ for each j, we may again apply Fatou's lemma as well as the additivity of the integral to obtain

$$\begin{aligned}
\int g\,d\mu - \int f\,d\mu &= \int (g - f)\,d\mu \\
&= \int \liminf_{j\to\infty} (g - f_j)\,d\mu \\
&\le \liminf_{j\to\infty} \int (g - f_j)\,d\mu \\
&= \int g\,d\mu - \limsup_{j\to\infty} \int f_j\,d\mu\,.
\end{aligned}$$

From this we may infer that

$$\limsup_{j\to\infty} \int f_j \, d\mu \le \int f \, d\mu \, . \tag{3}$$

Now, combining (2) and (3), we conclude that

$$\int f \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu \, . \qquad \square$$

Example: We again work with Lebesgue measure on the real line. Let $f_j(x) = \chi_{[j,j+1]}$ for $j = 1, 2, \ldots$. Then it is easy to see that there is no integrable function g with $|f_j| \leq g$ for all j. Thus we *may not* apply the dominated convergence theorem to conclude that

$$\lim_{j\to\infty} \int f_j \, d\mu = \int \lim_{j\to\infty} f_j \, d\mu \, ,$$

and in fact they are not equal.

Remark: In the next two examples we shall use a version of the Lebesgue Dominated Convergence Theorem that is a bit different from the formulation in the theorem. Namely, instead of a sequence of functions $f_j(x)$ as $j \to \infty$ we shall instead have a continuum of functions f_t parametrized by a parameter $t \in \mathbb{R}$ and consider $\lim_{t \to t_0} f_t(x)$. These two processes are in fact logically equivalent just because

$$\lim_{t \to t_0} f_t(x) = \ell \quad \text{if and only if}$$

$$\lim_{j \to \infty} f_{t_j}(x) = \ell \text{ for each sequence } t_j \to t_0 \,.$$

Example: We work with Lebesgue measure on the real line. Let us show that the function

$$F(t) = \int_{(0,\infty)} e^{-x} \cos(\pi t x) \, d\mu(x)$$

is continuous.

We intend to apply the dominated convergence theorem with

$$g(x) = \chi_{[0,\infty)}(x) \cdot e^{-x} = \begin{cases} e^{-x} & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \,. \end{cases}$$

Obviously g is measurable and $\chi_{[0,\infty)} \cdot e^{-x} \cdot \cos(\pi t x)$ is measurable for each t because e^{-x} is continuous and $\chi_{[0,\infty)}$ is measurable. We need to know that $\int_{\mathbb{R}} g \, d\mu < \infty$. But the monotone convergence theorem tells us that

$$\int_{\mathbb{R}} g \, d\mu = \lim_{j\to\infty} \int_{\mathbb{R}} \chi_{[-j,j]} \cdot g \, d\mu = \lim_{j\to\infty} \int_{\mathbb{R}} \chi_{[0,j]} \cdot e^{-x} \, d\mu(x)$$

$$= \lim_{j\to\infty} \int_0^j e^{-x} \, d\mu(x) \, .$$

The limit is of course 1 by the theory of the Riemann integral. So $\int_{\mathbb{R}} g \, d\mu < \infty$.

Now we may apply the dominated convergence theorem just because

$$|\chi_{[0,\infty)}(x) \cdot e^{-x} \cdot \cos(\pi t x)| \leq g(x)$$

for each $(x, t) \in \mathbb{R}^2$ (noting that $t \mapsto \chi_{[0,\infty)}(x) e^{-x} \cos(\pi t x)$ is continuous for each fixed x). We conclude that

$$\lim_{t \to t_0} F(t) = F(t_0) \, .$$

Example: A very standard operation in mathematical analysis is "differentiating under the integral sign." In this example we use the dominated convergence theorem to analyze and justify this operation.

Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

(a) $x \mapsto f^t(x) \equiv f(x,t)$ is measurable for each fixed $t \in \mathbb{R}$;

(b) $f^{t_0}(x) \equiv f(x,t_0)$ is integrable for some fixed $t_0 \in \mathbb{R}$;

(c) $\partial f(x,t)/\partial t$ exists for each (x,t).

Further suppose that there is an integrable function $g : \mathbb{R} \to \mathbb{R}$ so that

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x)$$

for each $x, t \in \mathbb{R}$.

Then the function $x \mapsto f(x,t)$ is integrable for each t and the function

$$F(t) = \int_{\mathbb{R}} f^t \, d\mu = \int_{\mathbb{R}} f(x,t) \, d\mu(x)$$

is differentiable with derivative

$$F'(t) = \frac{d}{dt} \int_{\mathbb{R}} f(x,t) \, d\mu(x) = \int_{\mathbb{R}} \frac{\partial}{\partial t} f(x,t) \, d\mu(x) \, . \qquad (1)$$

It is easy to see from equation (1) why this phenomenon is called "differentiation under the integral sign." Now we shall use the theory developed so far to see why this process is correct.

For each $t \neq t_0$, apply the mean value theorem to the function $t \mapsto f(x,t)$ to find a number c between t_0 and t so that

$$f(x,t) - f(x,t_0) = \left[\frac{\partial f}{\partial t}(x,c)\right] \cdot (t - t_0)\,.$$

It follows that

$$|f(x,t) - f(x,t_0)| \leq g(x) \cdot |t - t_0|$$

hence

$$|f(x,t)| \leq |f(x,t_0)| + g(x) \cdot |t - t_0|\,.$$

We conclude that

$$\begin{aligned}\int_{\mathbb{R}} |f(x,t)|\, d\mu(x) &\leq \int_{\mathbb{R}} (|f(x,t_0)| + g(x)\cdot|t-t_0|)\, d\mu(x) \\ &= \int |f(x,t_0)|\, d\mu(x) + |t-t_0| \int_{\mathbb{R}} g(x)\, d\mu(x)\,.\end{aligned}$$

This shows that the function $x \mapsto f(x,t)$ is integrable for each fixed t.

To establish the formula for F', consider any sequence $\{t_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} t_j = t$ and $t_j \neq t$ for each j. We claim that

$$\lim_{j\to\infty} \frac{F(t_j) - F(t)}{t_j - t} = \int_{\mathbb{R}} \frac{\partial}{\partial t} f(x,t)\, d\mu(x) \, . \qquad (2)$$

In fact we have

$$\frac{F(t_j) - F(t)}{t_j - t} = \int_{\mathbb{R}} \frac{f(x,t_j) - f(x,t)}{t_j - t}\, d\mu(x) \equiv \int_{\mathbb{R}} f_j(x,t)\, d\mu(x)$$

where

$$f_j(x,t) = \frac{f(x,t_j) - f(x,t)}{t_j - t} \, .$$

Observe that, for each fixed x, we know that

$$\lim_{j\to\infty} f_j(x,t) = \frac{\partial f}{\partial t}(x,t)$$

and hence (1) will follow from the dominated convergence theorem once we establish that $|f_j(x,t)| \le g(x)$ for each x.

But that claim follows from another application of the mean value theorem because there is a c' between t and t_j (with c' of course depending on x and t) such that

$$f_j(x,t) = \frac{f(x,t_j) - f(x,t)}{t_j - t} = \frac{\partial f}{\partial t}(x,c')\,.$$

Thus $|f_j(x,t)| \le g(x)$ for each x.