Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

In this lecture we are going to discuss infinite-dimensional vector spaces of functions. To this end we begin by reviewing the concept of vector space.

Definition: A *vector space* over the field $\mathbb{R}$ is a set V together with a binary operation of addition (denoted $+$) and a second operation of scalar multiplication (denoted $\cdot$) so that the following axioms are satisfied. Let $\mathsf{u}, \mathsf{v}, \mathsf{w}$ be elements of V (we call these *vectors*) and let a, b be scalars (i.e., real numbers). Then

(a) $\mathsf{u} + (\mathsf{v} + \mathsf{w}) = (\mathsf{u} + \mathsf{v}) + \mathsf{w}$;

(b) $\mathsf{u} + \mathsf{v} = \mathsf{v} + \mathsf{u}$;

(c) There exists an element $0 \in V$, called the *zero vector*, so that $\mathsf{v} + 0 = \mathsf{v}$ for any $\mathsf{v} \in V$.

(d) For every $\mathsf{v} \in V$, there is an element $-\mathsf{v} \in V$, called the *additive inverse* of v, so that $\mathsf{v} + (-\mathsf{v}) = 0$.

(e) $a \cdot (b \cdot \mathsf{v}) = (a \cdot b) \cdot \mathsf{v}$;

(f) $1 \cdot \mathsf{v} = \mathsf{v}$, where 1 denotes the number $1 \in \mathbb{R}$.

(g) $a \cdot (\mathsf{u} + \mathsf{v}) = a \cdot \mathsf{u} + a \cdot \mathsf{v}$;

(h) $(a + b) \cdot \mathsf{u} = a \cdot \mathsf{u} + b \cdot \mathsf{u}$.

Often in practice we omit the $\cdot$ when writing scalar multiplication.

Example: Of course $\mathbb{R}^N$ is a vector space over $\mathbb{R}$. The addition operation is

$$\langle a_1, a_2, \ldots, a_N \rangle + \langle b_1, b_2, \ldots, b_N \rangle = \langle a_1 + b_1, a_2 + b_2, \ldots, a_N + b_N \rangle .$$

The operation of scalar multiplication is

$$c \langle a_1, a_2, \ldots, a_N \rangle = \langle ca_1, ca_2, \ldots, ca_N \rangle .$$

The space ℓ^1 of all sequences $\{a_j\}_{j=1}^{\infty}$ such that $\sum_j |a_j| < \infty$ is a vector space. The operation of addition is

$$\{a_j\}_{j=1}^{\infty} + \{b_j\}_{j=1}^{\infty} = \{a_j + b_j\}_{j=1}^{\infty} .$$

The scalar multiplication operation is

$$c\{a_j\}_{j=1}^{\infty} = \{ca_j\}_{j=1}^{\infty} .$$

Definition: Let V be a vector space. A real-valued function N on V is said to be a *norm* if

(a) $N(\mathrm{v}) \geq 0$ for all $\mathrm{v} \in V$.

(b) $N(\mathrm{v}) = 0$ if and only if $\mathrm{v} = 0$.

(c) $N(\alpha \mathrm{v}) = |\alpha| N(\mathrm{v})$ for all $\mathrm{v} \in V$ and all real α.

(d) $N(\mathrm{u} + \mathrm{v}) \leq N(\mathrm{u}) + N(\mathrm{v})$ for all $\mathrm{u}, \mathrm{v} \in V$.

It is quite standard in many contexts to denote $N(\mathrm{v})$ by $\|\mathrm{v}\|$.

A *normed linear space* is a vector space equipped with a norm. If condition **(b)** for a norm is dropped then N is called a *seminorm.*

Example: If $V = \mathbb{R}^N$ and $\mathsf{v} = \langle x_1, x_2, \ldots, x_N \rangle \in V$, then we set

$$N(\mathsf{v}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}\,.$$

It is common to denote this norm on $\mathbb{R}^N$ by $\|\mathsf{v}\|$.

Another norm for $\mathbb{R}^N$ is given, for $p \geq 1$, by

$$\|\mathsf{v}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_N|^p)^{1/p}\ .$$

We shall not provide the details of the triangle inequality, but leave that matter as an exercise for you to think about.

The space ℓ^1, which we described above, has the norm

$$\|\{a_j\}\|_1 = \sum_j |a_j| \,.$$

The space X of functions f on $[0, 1]$ so that f' exists and is continuous forms a vector space. The expression

$$N(f) = \sup_{x \in [0,1]} |f'(x)|$$

is a seminorm on X. For if f is a (nonzero) constant function, then

$$N(f) = 0 \,.$$

Definition: Let $(X, mathcalX, \mu)$ be a measure space. Consider the space of all integrable functions. It is common to denote this space by $L^1(X, \mu)$ or just L^1. The norm on L^1 is

$$\|f\|_{L^1} = \int |f| \, d\mu \, .$$

In fact it is convenient to identify two integrable functions if they are equal almost everywhere. This is an equivalence relation on the set of all integrable functions. And we think, in practice, of L^1 as the collection of such equivalence classes.

Definition: Let $(X, mathcalX, \mu)$ be a measure space. Let $1 \leq p < \infty$. The space of all measurable functions f such that $|f|^p$ has finite integral is denoted by $L^p(X, \mu)$ or simply L^p. The norm on this space is

$$\|f\|_{L^p} = \int |f|^p \, d\mu^{1/p} \, .$$

We shall now prove a sequence of lemmas that will establish, among other things, the non-obvious fact that $\| \quad \|_{L^p}$ is actually a norm.

Proposition [Hölder's inequality]: *Let $f \in L^p$ and $g \in L^q$, where $1 < p < \infty$, $1 < q < \infty$, and $1/p + 1/q = 1$. Then $f \cdot g \in L^1$ and*

$$\|f \cdot g\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q} .$$

Proof: Let α be a real number with $0 < \alpha < 1$. Consider the function

$$\varphi(t) = \alpha t - t^\alpha$$

for $t \ge 0$. One may check that $\varphi'(t) < 0$ for $0 < t < 1$ and $\varphi'(t) > 0$ for $t > 1$. The mean value theorem then implies that $\varphi(t) \ge \varphi(1)$ and that $\varphi(t) = \varphi(1)$ if and only if $t = 1$. We conclude then that

$$t^\alpha \le \alpha t + (1 - \alpha) \quad \text{for} \quad t \ge 0 \, .$$

Let $a \ge 0$, $b > 0$, and set $t = a/b$ in this last inequality. Multiply through by b. The result is

$$a^\alpha \cdot b^{1-\alpha} \le \alpha a + (1 - \alpha) b \, . \tag{1}$$

Note that equality holds here if and only if $a = b$.

Now let $1 < p < \infty$ and $1/p + 1/q = 1$. Set $\alpha = 1/p$. If A, B are nonnegative numbers and if we set $a = A^{1/\alpha} = A^p$ and $B = B^{1/(1-\alpha)} = B^{p/(p-1)} = B^q$, then we may conclude from (1) that

$$AB \le \frac{A^p}{p} + \frac{B^q}{q} \tag{2}$$

and that equality holds if and only if $A^p = B^q$.

Now suppose that $f \in L^p$ and $g \in L^q$ and that $\|f\|_{L^p} \neq 0$ and $\|g\|_{L^q} \neq 0$. The product of these functions is certainly measurable and (2) with $A = |f(x)|/\|f\|_{L^p}$, $B = |g(x)|/\|g\|_{L^q}$ tells us that

$$\frac{|f(x) \cdot g(x)|}{\|f\|_{L^p}\|g\|_{L^q}} \le \frac{|f(x)|^p}{p\|f\|_{L^p}^p} + \frac{|g(x)|^q}{q\|g\|_{L^q}^q} \,.$$

Since both terms on the righthand side of this last inequality are integrable, it follow from an earlier corollary and the additivity of the integral that fg is integrable. On performing the integral we find that

$$\frac{\|fg\|_{L^1}}{\|f\|_{L^p}\|g\|_{L^q}} \leq \frac{1}{p} + \frac{1}{q} = 1\,.$$

This is Hölder's inequality. □