# Math 4121
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Figure: This is your instructor.

# The Lebesgue Integral

**Corollary [Cauchy-Schwarz-Bunyakovskii]:** *If $f$ and $g$ both belong to $L^2$, then $f \cdot g$ is integrable and*

$$\left| \int fg \, d\mu \right| \leq \int |fg| \; d\mu \leq \|f\|_{L^2} \cdot \|g\|_{L^2} \, .$$

*It is worth noting that the theorem is trivially true when $p = 1$ and $q = \infty$ or $p = \infty$ and $q = 1$. We shall treat the space $L^\infty$ in some detail below.*

The next result shows that the $L^p$ norms satisfy a triangle inequality.

**Proposition [Minkowski's inequality]:** *If the functions $f$ and $g$ both belong to $L^p$, $p \geq 1$, then $f + g$ also belongs to $L^p$ and*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p} . \tag{1}$$

**Proof:** The case $p = 1$ is easy, so we concentrate on $p > 1$. The sum $f + g$ is plainly measurable. Since

$$|f + g|^p \leq [2 \max\{|f|, |g|\}]^p \leq 2^p\{|f|^p + |g|^p\} ,$$

it follows from the corollary and the additivity of the integral that $f + g \in L^p$. Furthermore,

$$|f+g|^p = |f+g| \cdot |f+g|^{p-1} \leq |f| \cdot |f+g|^{p-1} + |g| \cdot |f+g|^{p-1} . \quad (2)$$

Since $f + g \in L^p$, we see that $|f + g|^p \in L^1$. Furthermore, since $p = (p-1)q$, it follows that $|f+g|^{p-1} \in L^q$. Thus we can apply Hölder's inequality to conclude that

$$\begin{aligned} \int |f||f+g|^{p-1}\, d\mu &\leq \|f\|_{L^p} \cdot \left\{ \int |f+g|^{(p-1)q}\, d\mu \right\}^{1/q} \\ &= \|f\|_{Lp} \cdot \|f+g\|_{L^p}^{p/q} . \end{aligned}$$

If we treat the second term on the right in (2) similarly, the result is

$$\begin{aligned} \|f+g\|_{L^p}^p &\leq \|f\|_{L^p} \cdot \|f+g\|_{L^p}^{p/q} + \|g\|_{L^p}\|f+g\|_{L^p}^{p/q} \\ &= (\|f\|_{L^p} + \|g\|_{L^p}) \cdot \|f+g\|_{L^p}^{p/q} . \end{aligned}$$

If $M = \|f + g\|_{L^p} = 0$, then equation (1) is trivial. If instead $M \neq 0$, then we can divide the last inequality by $M^{p/q}$. Since $p - p/q = 1$, Minkowski's inequality results. □

A *Banach space* is a normed linear space that is complete. This means that any Cauchy sequence has a limit *in that space*. The theory of Banach spaces is rich and fertile. It is a powerful tool in mathematical analysis. Our next task is to show that the $L^p$ spaces are Banach spaces.

**Definition:** A sequence $\{f_j\} \subseteq L^p$ is said to be *convergent* to $f \in L^p$ if, for every $\epsilon > 0$, there is a number $J > 0$ so that if $j > J$, then $\|f_j - f\|_{L^p} < \epsilon$.

The sequence $\{f_j\}$ in $L^p$ is a *Cauchy sequence* if, for every $\epsilon > 0$, there is a number $J > 0$ so that if $j, k > J$, then $\|f_j - f_k\|_{L^p} < \epsilon$.

**Definition:** The space $L^p$ is *complete* if every Cauchy sequence in $L^p$ converges to an element $f \in L^p$.

**Lemma:** *If the sequence* $\{f_j\} \subseteq \text{Ł}^p$ *converges to* $f \in L^p$, *then the sequence is Cauchy.*

**Proof:** Let $\epsilon > 0$. Choose $J > 0$ so that $j > J$ implies that $\|f_j - f\|_{L^p} < \epsilon/2$. Now let $j, k > J$. Then

$$\|f_j - f_k\|_{L^p} \le \|f_j - f\|_{L^p} + \|f - f_k\|_{L^p} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

It follows that the sequence $\{f_j\}$ is Cauchy. □

**Theorem:** *Let* $1 \leq p < \infty$*. Then the space* $L^p$ *is a complete, normed, linear space.*

**Proof:** Let $\{f_j\}$ be a Cauchy sequence in the $L^p$ norm. Our job is to show that there exists an $f \in L^p$ so that $f_j \to f$ in the $L^p$ norm.

Our hypothesis tells us that, if $\epsilon > 0$, then there is a number $J > 0$ so that if $j, k > J$, then

$$\int |f_j - f_k|^p \, d\mu = \|f_j - f_k\|_{L^p}^p < \epsilon^p \,. \tag{1}$$

Thus there exists a subsequence $\{f_{j_k}\}$ such that

$$\|f_{j_{k+1}} - f_{j_k}\|_{L^p} < 2^{-k}$$

for $k \in$ . Define

$$g(x) = |f_{j_1}(x)| + \sum_{k=1}^{\infty} |f_{j_{k+1}}(x) - f_{j_k}(x)| \,.$$

Thus $g$ is measurable, nonnegative, and $p$th power integrable.

Indeed, by Fatou's lemma,

$$\int |g|^p \, d\mu \leq \liminf_{n\to\infty} \int \left\{ |f_{j_1}| + \sum_{j=1}^{n} |f_{j_{k+1}} - f_{j_k}| \right\}^p \, d\mu \, .$$

Taking $p$th roots of both sides and applying Minkowski's inequality we find that

$$\begin{aligned} \left\{ \int |g|^p \, d\mu \right\}^{1/p} &\leq \liminf_{n\to\infty} \left\{ \|f_{j_1}\|_{L^p} + \sum_{k=1}^{n} \|f_{j_{k+1}} - f_{j_k}\|_{L^p} \right\} \\ &\leq \|f_{j_1}\|_{L^p} + 1 \, . \end{aligned}$$

Thus, if $E = \{x \in \ : g(x) < +\infty\}$, then $E \in \mathcal{X}$ and $\mu(X \setminus E) = 0$. Therefore the series in defining $g$ converges almost everywhere and $g\chi_E$ belongs to $L^p$.

We now define $f$ on $\mathbb{R}$ by

$$f(x) = \begin{cases} f_{j_1}(x) + \sum_{k=1}^{\infty} \{f_{j_{k+1}}(x) - f_{j_k}(x)\} & \text{if} \quad x \in E\,, \\ 0 & \text{if} \quad x \notin E\,. \end{cases}$$

Because the series in this definition is absolutely convergent almost everywhere, it follows that it is convergent almost everywhere. Since

$$|f_{j_k}| \le |f_{j_1}| + \sum_{\ell=1}^{k} |f_{j_{\ell+1}} - f_{j_\ell}| \le g\,,$$

hence

$$|f_{j_k}|^p \le g^p\,,$$

and since $\{f_{j_k}\}$ converges almost everywhere to $f$, the dominated convergence theorem now tells us that $f \in L^p$. Also, since $|f - f_{j_k}|^p \le 2^p \cdot g^p$, we conclude from the dominated convergence theorem that $0 = \lim_{k\to\infty} \|f - f_{j_k}\|_{L^p}$ so that $\{f_{j_k}\}$ converges in the $L^p$ norm to $f$.

Because of (1), if $\epsilon > 0$ and $m > J$ as at the start of the proof, and if $k$ is sufficiently large, then

$$\int |f_m - f_{j_k}|^p \, d\mu < \epsilon^p \, .$$

Applying Fatou's lemma, we now conclude that

$$\int |f_m - f|^p \, d\mu \le \liminf_{k \to \infty} \int |f_m - f_{j_k}|^p \, d\mu \le \epsilon^p$$

whenever $m > J$. This proves that the sequence $\{f_m\}$ converges to $f$ in the $L^p$ norm. $\square$

Thus we now know that each $L^p$ space, $1 \leq p < \infty$, is a Banach space.