Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

The Case $p = \infty$

When discussing L^p spaces above, we always restricted p to be less than $+\infty$. But in fact L^∞ is an interesting and important space in its own right. We consider it now.

First a little review. Let $f : X \to \mathbb{R}$ be a function. A real number a is called an *upper bound* for f if $f(x) \le a$ for all x. A convenient way of saying this is that

$$f^{-1}((a,\infty)) = \{x \in X : f(x) > a\} = \emptyset \, .$$

Set

$$U_f = \{a \in \mathbb{R} : f^{-1}((a,\infty)) = \emptyset\}$$

and define

$$\sup f = \inf U_f \, .$$

This defines the supremum or least upper bound of the function f.

Now, by analogy, consider a measure space $(X, \mathcal{X}, \mu)$. Let f be a measurable function on X. A real number a is called an *essential upper bound* for f if the set $f^{-1}((a, \infty))$ has measure zero. In other words, a is an essential upper bound if $f(x) \leq a$ for almost all $x \in X$. As we did in the last paragraph, let

$$U_f^{\text{ess}} = \{a \in \mathbb{R} : \mu(f^{-1}((a, \infty))) = 0\}$$

be the set of all essential upper bounds. We define the *essential supremum* of f to be

$$\operatorname{ess\,sup} f = \inf U_f^{\text{ess}}$$

if $U_f^{\text{ess}} \neq \emptyset$ and $\operatorname{ess\,sup} f = +\infty$ otherwise.

Of course the *essential infimum* of f is defined in just the same way. A function $f : X \to \mathbb{R}$ is said to be *essentially bounded* if it has a finite essential supremum and a finite essential infimum.

Example: We work with Lebesgue measure as usual. Let

$$f(x) = \begin{cases} 6 & \text{if} \quad x = 3\,, \\ -5 & \text{if} \quad x = -3\,, \\ 1 & \text{if} \quad x \neq 3, -3\,. \end{cases}$$

The supremum of this function is 6 and the infimum is -5. But these values are assumed only on a subset of the domain that has measure 0. Off that set of measure 0, the function is constantly equal to 1. Thus the essential infimum and the essential supremum of f are both 1.

Now define

$$g(x) = \begin{cases} x^5 & \text{if} \quad x \in \mathbb{Q}\,, \\ \arctan 2x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}\,. \end{cases}$$

This function is unbounded, both above and below. So its supremum is $+\infty$ and its infimum is $-\infty$. But the unboundedness is based on values the function takes on domain elements in the rational numbers (which is a set of measure 0). On the irrational numbers, which is a set of full measure, the function is bounded above by $\pi/2$ and below by $-\pi/2$. Thus the essential supremum is $\pi/2$ and the essential infimum is $-\pi/2$.

Definition: Let $(X, \mathcal{X}, \mu)$ be a measure space. The space $L^\infty(X, \mu)$ or just L^∞ is defined to be the collection of essentially bounded functions. The norm on L^∞ is

$$\|f\|_{L^\infty} = \max\{|\text{ess sup}\, f|, |\text{ess inf}\, f|\} .$$

Theorem: The space L^∞ is a Banach space.

Proof: That L^∞ is a linear space is routine, and we leave the details for the reader.

The interesting part is to check that L^∞ is complete. So let $\{f_j\}$ be a Cauchy sequence in L^∞. Let $E \subseteq X$ be a set of measure 0 such that $|f_j(x)| \leq \|f_j\|_{L^\infty}$ for $x \not\in E$ and $j = 1, 2, \dots$ and also so that $|f_j(x) - f_k(x)| \leq \|f_j - f_k\|_{L^\infty}$ for all $x \not\in E$, $j, k = 1, 2, \dots$ (note that the set E is obtained by taking the countable union of sets of measure zero coming from different values of j and k). Then the sequence $\{f_j\}$ is uniformly convergent on $X \setminus E$. We let

$$f(x) = \begin{cases} \lim_{j\to\infty} f_j(x) & \text{if} \quad x \not\in E\,, \\ 0 & \text{if} \quad x \in E\,. \end{cases}$$

It follows that f is measurable. It is easy to check that $\lim_{j\to\infty} \|f_j - f\|_{L^\infty} \to 0$. Thus L^∞ is complete. $\square$

For this and the next three lectures we shall concentrate on presenting the idea of Lebesgue measure from a different point of view. This is using the idea of outer measure. In fact outer measures are quite intuitive, and you may find this approach to be appealing. This is the approach that many texts use.

The approach that we present here works very well in $\mathbb{R}^N$ for all positive, integer values of N. But, in order to keep things simple, we shall concentrate on $\mathbb{R}^1$.

Definition: Let $E \subseteq \mathbb{R}$. We define the *Lebesgue outer measure* $m^*(E)$ of E to be

$$m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \ell(I_j) \right\} ,$$

where the infimum is taken over all sequences $\{I_j\}$ of open intervals in $\mathbb{R}$ that cover E in the sense that

$$E \subseteq \bigcup_{j=1}^{\infty} I_j \, .$$

Of course ℓ here denotes the ordinary notion of length of an interval.

Remark: Outer measure has these properties.

1. Since the intervals $I_j = (j-1, j+1)$ cover all of $\mathbb{R}$, they certainly cover any subset E of $\mathbb{R}$. So the infimum above is *not* over the empty set. Clearly $m^*(E) \geq 0$. It is also certainly possible for $m^*(E) = +\infty$.
2. The terms $\ell(I_j)$ are all nonnegative. So the series

$$\sum_{j=1}^{\infty} \ell(I_j)$$

is either **(i)** absolutely convergent (in which case the value of the sum does not depend on the order of the intervals) or **(ii)** divergent, in which case the sum takes the value $+\infty$.

Remark:

(a) It is most common to use open intervals in the definition of outer measure. But we could just as easily use closed intervals, and the resulting theory would be the same.

(b) It is also possible to use half-open intervals to define the outer measure. But there is no compelling reason to do so.

(c) Fix a number $\delta > 0$. We could define the outer measure using intervals that have length not exceeding δ. The same theory would result.

Theorem: The outer measure function m^* satisfies:

(a) $0 \leq m^*(E) \leq +\infty$ for all $E \subseteq \mathbb{R}$.

(b) $m^*(\emptyset) = 0$.

(c) If $E \subseteq F$, then $m^*(E) \leq m^*(F)$.

(d) If $\{E_j\}$ are countably many subsets of $\mathbb{R}$, then

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j) .$$

Remark: Notice that we did *not* in this theorem specify countable additivity for outer measure. The simple reason is that outer measure is *not* countably additive when it is applied to *all* subsets of $\mathbb{R}$. We must find a criterion that allows us to restrict attention to a particular collection of subsets of $\mathbb{R}$. This is what we do below.

Proof of the Theorem:

(a) The first property is obvious from our earlier discussion.

(b) This property is also obvious if we take each I_j to be the empty set.

(c) If $\{I_j\}$ is a sequence of intervals whose union contain F then that union also contains E. That gives the result.

(d) It suffices to prove the result when $m^*(E_k) < \infty$ for each k. Let $\epsilon > 0$ and, for each $k \in \mathbb{N}$, choose a sequence $\{I_j^k\}_{j=1}^{\infty}$ of intervals such that

$$E_k \subseteq \bigcup_{j=1}^{\infty} I_j^k \quad \text{and} \quad \sum_{j=1}^{\infty} \ell(I_j^k) \leq m^*(E_k) + \frac{\epsilon}{2^k} \, .$$

Since $\{I_j^k : j, k \in \mathbb{N}\}$ is a countable family of intervals that covers the set $\cup_{k=1}^{\infty} E_k$, we see from the definition of m^* that

$$\begin{aligned} m^*\left(\bigcup_{\ell=1}^{\infty} E_\ell\right) &\leq \sum_{j,k=1}^{\infty} \ell(I_j^k) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_j^k) \\ &\leq \sum_{k=1}^{\infty}(m^*(E_k) + \epsilon/2^k) \\ &= \sum_{k=1}^{\infty} m^*(E_k) + \epsilon\,. \end{aligned}$$

Since $m^*(E) \geq 0$ for any set $E \subseteq \mathbb{R}$, the change from a double sum to an iterated sum is justified. Now, since $\epsilon > 0$ is arbitrary, the proof of property **(d)** is complete.

□

Property **(d)** of the theorem is commonly referred to as the *countable subadditivity* property. One consequence of **(d)** is that, if A and B are *disjoint* sets then

$$m^*(A \cup B) \leq m^*(A) + m^*(B) \,.$$

From previous experience, one might expect to have equality in this last displayed equation. However such a property does not hold. We can show that, if there is a positive distance between A and B, then equality obtains. But without such an artificial hypothesis, we will find that we need to restrict attention to a special class of sets in order to get equality.

Proposition: Let A and B be disjoint subsets of $\mathbb{R}$ with

$$\mathrm{dist}(A, B) \equiv \inf\{|a - b| : a \in A, b \in B\} > 0 .$$

Then

$$m^*(A \cup B) = m^*(A) + m^*(B) .$$

Proof: We saw in the previous displayed equation that the result is true with $\leq$ replacing $=$. Thus it suffices to prove the reverse inequality under the hypothesis that $m^*(A \cup B) < +\infty$ and $\delta = \text{dist}(A, B) > 0$.

Let $\epsilon > 0$. Let $\{I_j\}$ be a covering of $A \cup B$ by open intervals such that

$$\sum_{j=1}^{\infty} \ell(I_j) \leq m^*(A \cup B) + \epsilon .$$

As previously noted, we may assume that the intervals I_j each have length less than δ. Thus none of the I_j can contain both points of A and points of B.

As a result, we can divide the intervals I_j into three classes:

(i) The intervals J_j that contain points of A;

(ii) The intervals K_j that contain points of B;

(iii) The intervals H_j that contain neither points in A nor points in B.

Thus we have

$$m^*(A) \le \sum_j \ell(J_j) \quad \text{and} \quad m^*(B) \le \sum_j \ell(K_j) \, .$$

From this it follows that

$$\begin{aligned} m^*(A) + m^*(B) &\le \sum_j \ell(J_j) + \sum_j \ell(K_j) + \sum_j \ell(H_j) \\ &= \sum_j \ell(I_j) \\ &\le m^*(A \cup B) + \epsilon \, . \end{aligned}$$

We conclude that $m^*(A) + m^*(B) \le m^*(A \cup B) + \epsilon$. Since $\epsilon > 0$ is arbitrarily small, we conclude that $m^*(A) + m^*(B) \le m^*(A \cup B)$, as was to be proved. □

Next we show that, at least for intervals, the outer measure m^* gives no surprises.

Proposition: If I is any bounded, open interval, then $m^*(I) = \ell(I)$.

Proof: Since the sequence $\{I, \emptyset, , \ldots\}$ is a covering of I, it follows that

$$m^*(I) \le \ell(I) + \ell(\emptyset) + \ell(\emptyset) + \cdots = \ell(I) + 0 + 0 + \cdots = \ell(I) .$$

That establishes one direction.

For the opposite inequality, let $\epsilon > 0$ and let $\{I_j\}_{j=1}^{\infty}$ be a covering of I by open intervals so that

$$\sum_{j=1}^{\infty} \ell(I_j) \le m^*(I) + \epsilon .$$

Let J be a closed, bounded interval contained in I and so that $\ell(I) - \epsilon < \ell(J)$. The Heine-Borel theorem then tells us that there is an $m \in \mathbb{N}$ such that $J \subseteq \cup_{j=1}^{m} I_j$. See the figure.

Figure: An open covering.

Of course the intervals I_j will in general have some overlap. Let $p_1, p_2, \ldots, p_k$ be the endpoints of the I_j in the natural order in which they occur on the real line. Consider any closed interval which has endpoints some sequential pair p_j, p_{j+1} and which is a subset of one of the I_j. Call those closed intervals $K_1, K_2, \ldots, K_p$. Likewise let $J_1, J_2, \ldots, J_q$ be those closed intervals into which J is divided by the p_j. Then we have

$$\begin{aligned} \ell(J) &= \sum_{k=1}^{q} \ell(J_k) \\ &\le \sum_{\ell=1}^{p} \ell(K_\ell) \\ &\le \sum_{r=1}^{m} \ell(I_r) \\ &\le m^*(I) + \epsilon \,. \end{aligned}$$

We see that $\ell(I) \le \ell(J) + \epsilon \le m^*(I) + 2\epsilon$. Since $\epsilon > 0$ is arbitrarily small, we conclude that $\ell(I) \le m^*(I)$.

We conclude then that $\ell(I) = m^*(I)$, as was to be proved. $\square$

Remark: A similar result can be proved for closed intervals, or for half-open intervals. We leave the details for the interested reader.

It is an easily established fact, and we leave the details for the reader, that outer measure is translation invariant. That is to say, if $E \subseteq \mathbb{R}$ is a set and $a \in \mathbb{R}$ and $E_a \equiv \{e + a : e \in E\}$, then $m^*(E) = m^*(E_a)$.