# Math 4121
# February 22, 2021 Lecture

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Figure: This is your instructor.

# The Lebesgue Integral

# A New Look at Measurable Sets

# Identifying Measurable Sets

In our first go-around, we decided what a measurable set was by fiat. More precisely, we defined the concept of $\sigma$-algebra and then declared, "This is the $\sigma$-algebra of measurable sets." Now, in our new development, the point of view is different. Here we will understand measurable sets by more of a discovery method. And the measurable sets defined with our new technique will have all the desirable properties of a measure—including countable additivity.

We begin by recalling the definition of $\sigma$-algebra. We shall make considerable use of this concept in our current discussion. Compare with our consideration of $\sigma$-algebras in earlier lectures.

**Definition:** Let $X$ be any set. Then a family $\mathcal{X}$ of subsets of $X$ is said to be a *$\sigma$-algebra* in $X$ if these conditions are satisfied:

**(i)** $\emptyset$ and $X$ both belong to $\mathcal{X}$;

**(ii)** if $E \in \mathcal{X}$, then the complement $X \setminus E$ also belongs to $\mathcal{X}$;

**(iii)** if $\{E_j\}_{j=1}^{\infty}$ is a sequence of sets in $\mathcal{X}$, then the union $\cup_{j=1}^{\infty} E_j$ also belongs to $\mathcal{X}$.

**Remark:** We note the following points.

**(a)** If $\mathcal{X}$ is a $\sigma$-algebra of subsets of $X$, then the intersection of a sequence of sets in $\mathcal{X}$ also belongs to $\mathcal{X}$. This is an immediate consequence of de Morgan's laws.

**(b)** If $X$ is any set, then $\{\emptyset, X\}$ is a trivial example of a $\sigma$-algebra.

**(c)** If $X$ is any set and $E \subseteq X$, then $\mathcal{X} \equiv \{\emptyset, E, {}^cE, X\}$ is a $\sigma$-algebra.

**(d)** If $X$ is any set, then the power set $\mathcal{P}(X)$ is a $\sigma$-algebra.

**(e)** If $X$ is any set and if $\mathcal{X}_1$ and $\mathcal{X}_2$ are $\sigma$-algebras of subsets of $X$, then $\mathcal{X}_1 \cap \mathcal{X}_2$ is also a $\sigma$-algebra.

**Definition:** Let $X$ be a set and let $\mathcal{X}$ be a $\sigma$-algebra of subsets of $X$. Then an $\widehat{\mathbb{R}}$-valued function $\mu$ with domain $\mathcal{X}$ is said to be a *measure* provided that

**(i)** $\mu(\emptyset) = 0$;

**(ii)** $0 \le \mu(E) \le +\infty$ for all $E \in \mathcal{X}$;

**(iii)** if $\{E_j\}_{j=1}^{\infty}$ is a sequence of sets in $\mathcal{X}$ that are pairwise disjoint, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)\,. \tag{1}$$

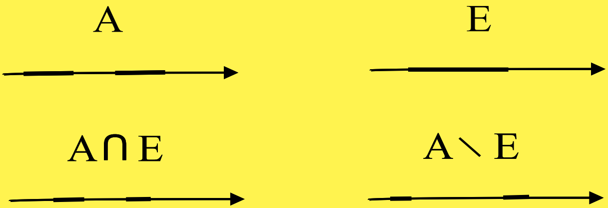
**Example:** If $X = \mathbb{N}$ and $\mathcal{X} = \{\text{all subsets of } X\}$, then define $\mu(E)$ to be the number of elements in $E$ if $E$ is a finite set and to be $+\infty$ if $E$ is an infinite set. Then $\mu$ is a measure on $\mathcal{X}$. It is called the *counting measure* on $\mathbb{N}$.

Now we have reached a crucial juncture. We are going to explicitly define the criterion for measurability of a set.

**Definition:** Let $m^*$ be the outer measure defined on all subsets of $\mathbb{R}$. A set $E \subseteq \mathbb{R}$ is said to satisfy the *Carathéodory condition* in case

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) = m^*(A \cap E) + m^*(A \cap {}^c E)$$

for all subsets $A \subseteq \mathbb{R}$. See the figure. The collection of all such sets will be denoted by $\mathcal{L}$.


A
E
$A \cap E$
$A \setminus E$