Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Identifying Measurable Sets

We see that a set E satisfies Carathéodory's condition if E and its complement split every set A in an additive fashion. The sets that satisfy this condition are the ones that we shall think of as the *measurable sets*. The next result shows that the task of checking measurability can be simplified a bit.

Lemma: A set E satisfies the Carathéodory condition if and only if, for each set A with $m^*(A) < \infty$, we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E) .$$

Proof: Since $A \cap E$ and $A \setminus E$ are disjoint and have union A, we see from part **(d)** of the theorem that we always have the inequality

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E) .$$

Thus, if the single inequality is satisfied, then so is the equality.

Observe in passing that our inequality is trivial in case $m^*(A) = +\infty$. So it is only necessary to think about the case $m^*(A) < \infty$. □

Theorem [Carathéodory]: Let m^* be the outer measure defined earlier. Then the set $\mathcal{L}$ of all subsets of $\mathbb{R}$ that satisfy the Carathéodory condition is a σ-algebra of subsets of $\mathbb{R}$. Furthermore, the restriction of m^* to $\mathcal{L}$ is a measure on $\mathcal{L}$.

Proof: It is clear that the empty set satisfies our condition. Also, if E satisfies the condition, then so does its complement ${}^cE$. Therefore the family of sets that satisfy the Carathéodory condition satisfies properties **(i)** and **(ii)** of our definition.

We next show that, if E and F satisfy our condition, then so does $E \cap F$. This is the case because, since $E \in \mathcal{L}$, we have that

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

for any $A \subseteq \mathbb{R}$. Because $F \in \mathcal{L}$, we have

$$m^*(A \cap E) = m^*(A \cap E \cap F) + m^*(A \cap E \cap {}^cF) .$$

Thus

$$m^*(A) = m^*(A \cap E \cap F) + m^*(A \cap E \cap {}^cF) + m^*(A \cap {}^cE) .$$

But, since $E \in \mathcal{L}$, we have in addition that

$$\begin{aligned} m^*(A \cap {}^c(E \cap F)) &= m^*(A \cap {}^c(E \cap F) \cap E) + m^*(A \cap {}^c(E \cap F) \cap {}^cE) \\ &= m^*(A \cap {}^cF \cap E) + m^*(A \cap {}^cE) \, . \end{aligned}$$

Thus we see that

$$m^*(A) = m^*(A \cap (E \cap F)) + m^*(A \cap {}^c(E \cap F))$$

for all sets A. Thus $E \cap F$ belongs to $\mathcal{L}$.

Since $\mathcal{L}$ contains the complements of sets in $\mathcal{L}$, we see from de Morgan's laws that, if $E, F \in \mathcal{L}$, then $E \cup F \in \mathcal{L}$. Furthermore, if $E \cap F = \emptyset$, then it follows from the fact that E satisfies our condition with A replaced by $A \cap (E \cup F)$ and $F = F \cap {}^cE$ that

$$\begin{aligned} m^*(A \cap (E \cup F)) &= m^*(A \cap (E \cup F) \cap E) + m^*(A \cap (E \cup F) \cap {}^cE) \\ &= m^*(A \cap E) + m^*(A \cap F) . \end{aligned}$$

By induction we then see that, if $E_1, E_2, \ldots, E_k$ belongs to $\mathcal{L}$ and are pairwise disjoint, then $E_1 \cup E_2 \cup \cdots \cup E_k$ belongs to $\mathcal{L}$ and

$$m^*(A \cap (E_1 \cup E_2 \cup \cdots \cup E_k)) = m^*(A \cap E_1) + \cdots + (A \cap E_k)$$

for all $A \subseteq \mathbb{R}$.

Now our task is to show that $\mathcal{L}$ is a σ-algebra and that m^* is countably additive on $\mathcal{L}$. To this end, let $\{E_j\}$ be a pairwise disjoint sequence of sets in $\mathcal{L}$ and let $E = \cup_{j=1}^{\infty} E_j$. Certainly $F_n \equiv \cup_{j=1}^{n} E_j$ belongs to $\mathcal{L}$ for all $n \in \mathbb{N}$. Further, if $A \subseteq \mathbb{R}$, then

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap {}^c F_n) \\ &= m^*\left(\bigcup_{j=1}^{n} A \cap E_j\right) + m^*(A \cap {}^c F_n) \\ &= \sum_{j=1}^{n} m^*(A \cap E_j) + m^*(A \cap {}^c F_n) \,. \end{aligned}$$

Since $F_n \subseteq E$, we see that $A \cap {}^c F_n \supseteq A \cap {}^c E$ for all $n \in \mathbb{N}$. Hence

$$m^*(A) \geq \sum_{j=1}^{n} m^*(A \cap E_j) + m^*(A \cap {}^c E) \,.$$

This inequality implies that

$$m^*(A) \geq \sum_{j=1}^{\infty} m^*(A \cap E_j) + m^*(A \cap {}^c E) \,.$$

It follows now from the countable subadditivity of m^* that

$$m^*(A \cap E) = m^*\left(\bigcup_{j=1}^{\infty} A \cap E_j\right) \leq \sum_{j=1}^{\infty} m^*(A \cap E_j) \,.$$

Thus we have that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap {}^cE) \, .$$

This, in view of the above lemma, implies that $E \in \mathcal{L}$. Hence $\mathcal{L}$ is a σ-algebra. What is more, if we take $A = E$ in our theorems, then we have

$$m^*(E) = \sum_{j=1}^{\infty} m^*(E_j) \, .$$

This shows that m^* is countably additive on $\mathcal{L}$. $\square$

Definition: If m^* is the outer measure defined in the last chapter, then the σ-algebra $\mathcal{L}$ of subsets of $\mathbb{R}$ that satisfy the Carathéodory condition is called the *Lebesgue σ-algebra of* $\mathbb{R}$. A set $E \in \mathcal{L}$ is called a *Lebesgue measurable subset of* $\mathbb{R}$ or, briefly, a *measurable subset of* $\mathbb{R}$. The restriction of m^* to $\mathcal{L}$, which we now call m, is called the *Lebesgue measure* on $\mathbb{R}$.

Remark: Since m is the restriction of m^* to the σ-algebra $\mathcal{L}$, we know that $m(E) = m^*(E)$ for every $E \in \mathcal{L}$. Most of the time, when we know that a set E is measurable, we shall write $m(E)$ instead of $m^*(E)$.

Proposition: If I is an interval in $\mathbb{R}$, then I is measurable and $m(I) = \ell(I)$.

Proof: We shall give the proof for an open interval and leave the cases of the other intervals to the reader.

We saw in the lemma that it is enough to show that, if $A \subseteq \mathbb{R}$ is such that $m^*(A) < +\infty$, then

$$m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I) .$$

Let $n \in \mathbb{N}$ and let $I_n = \{x \in I : \text{dist}(x, {}^c I) > 1/n\}$. Hence $I_n \subseteq I$. Also, since $I \setminus I_n$ lies in the union of two cells each of which has side length $1/n$, then $m^*(I \setminus I_n) \to 0$ as $n \to \infty$.

Notice that $A \supseteq (A \cap I_n) \cup (A \setminus I)$ and that $\text{dist}(A \cap I_n, A \setminus I) \geq 1/n$. Thus we have from the proposition that

$$\begin{aligned} m^*(A) &\geq m^*((A \cap I_n) \cup (A \setminus I)) \\ &= m^*(A \cap I_n) + m^*(A \setminus I) . \end{aligned} \tag{1}$$

But we also know that

$$A \cap I = (A \cap I_n) \cup (A \cap (I \setminus I_n)) .$$

Thus it follows from the subadditivity and the monotone character of m^* that

$$m^*(A \cap I_n) \leq m^*(A \cap I) \leq m^*(A \cap I_n) + m^*(A \cap (I \setminus I_n)) .$$

Thus we have

$$m^*(A \cap I) = \lim_{n\to\infty} m^*(A \cap I_n)\,.$$

So, taking the limit in (1), we have

$$m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I)\,.$$

This shows, by the lemma, that I is a measurable set.

It is an easy calculation to see that $m(I) = \ell(I)$. Certainly any open covering of I has total length at least $\ell(I)$. And it is a simple matter to produce coverings with total length less than $\ell(I) + \epsilon$ for any $\epsilon > 0$. That does it. $\square$

What we have accomplished thus far is both interesting and valuable. Namely, we have a measure m defined on a σ-algebra $\mathcal{L}$ of sets that agrees with the length function ℓ—and ℓ was originally defined only for intervals. Thus we now know, for $\mathcal{B}$ the Borel sets, that $\mathcal{B} \subseteq \mathcal{L}$ and we have succeeded in extending ℓ from $\mathcal{B}$ to the notably larger collection of sets $\mathcal{L}$. We will spend some time seeing just how large $\mathcal{L}$ is.

It is conceivable that there is another measure defined on $\mathcal{L}$ that also agrees with ℓ on intervals. We now show that this is not the case.

Theorem: If μ is a measure defined on the σ-algebra $\mathcal{L}$ that satisfies $\mu(I) = \ell(I)$ for all open intervals I, then $\mu = m$.

Proof: For $n \in \mathbb{N}$, let $I_n = (-n, n)$. Let $E \in \mathcal{L}$ be any set with $E \subseteq I_n$ and let $\{J_k\}$ be a sequence of open intervals such that $E \subseteq \cup_{k=1}^{\infty} J_k$. Since μ is a measure and $\mu(J_k) = \ell(J_k)$ for all $k \in \mathbb{N}$, we see that

$$\mu(E) \leq \mu\left(\bigcup_{k=1}^{\infty} J_k\right) \leq \sum_{k=1}^{\infty} \mu(J_k) = \sum_{k=1}^{\infty} \ell(J_k)\,.$$

Thus we have $\mu(E) \leq m^*(E) = m(E)$ for all measurable sets $E \subseteq I_n$.

Since μ and m are additive, we have

$$\mu(E) + \mu(I_n \setminus E) = \mu(I_n) = m(I_n) = m(E) + m(I_n \setminus E)\,.$$

Because all these terms are finite and $\mu(E) \leq m(E)$ and $\mu(I_n \setminus E) \leq m(I_n \setminus E)$, we may conclude that $\mu(E) = m(E)$ for all measurable sets $E \subseteq I_n$.

Note that an arbitrary measurable set E can be written as the union of a pairwise disjoint sequence $\{E_j\}$ of bounded sets, defined by

$$E_1 = E \cap I_1 \quad , \quad E_j = E \cap (I_j \setminus I_{j-1}) \ , \quad \text{for } j > 1 \, .$$

Since $\mu(E_j) = m(E_j)$ for all $j \in \mathbb{N}$, it follows that

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} m(E_j) = m(E) \, .$$

In conclusion, μ and m agree on all measurable sets. □

We wrap up this lecture with two useful and intuitively obvious facts about Lebesgue measure.

Proposition: If E and F are Lebesgue measurable sets and if $E \subseteq F$, then $m(E) \leq m(F)$. If in addition $m(E) < +\infty$, then $m(F \setminus E) = m(F) - m(E)$.

Proof: Since m is additive, and since we know that $F = E \cup (F \setminus E)$ and $E \cap (F \setminus E) = \emptyset$, then we have

$$m(F) = m(E) + m(F \setminus E) \, .$$

Because $m(F \setminus E) \geq 0$, we conclude that $m(F) \geq m(E)$. If $m(E) < +\infty$, then we can substract $m(E)$ from both sides of the above equation to obtain the second assertion. $\square$

Theorem:

(a) If $\{E_j\}$ is an increasing sequence of Lebesgue measurable sets, then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} m(E_n)\,.$$

(b) If $\{F_j\}$ is a decreasing sequence of Lebesgue measurable sets and if $m(F_1) < \infty$, then

$$m\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{n\to\infty} m(F_n)\,.$$

Proof:

(a) If $m(E_k) = +\infty$ for some $k \in \mathbb{N}$, then both sides of our equation are equal to $+\infty$. Thus we may assume that $m(E_j) < +\infty$ for all $j \in \mathbb{N}$. Let $A_1 = E_1$ and $A_j = E_j \setminus E_{j-1}$ for $j > 1$. Then $\{A_j\}$ is a pairwise disjoint sequence of measurable sets such that

$$E_j = \bigcup_{n=1}^{j} A_n \quad \text{and} \quad \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} A_j \,.$$

Since m is countably additive, we see that

$$\begin{aligned} m\left(\bigcup_{j=1}^{\infty} E_j\right) &= m\left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} m(A_n) \\ &= \lim_{p\to\infty} \left(\sum_{n=1}^{p} m(A_n)\right) . \end{aligned}$$

By the proposition, we see that

$$m(A_n) = m(E_n) - m(E_{n-1})$$

for $n > 1$. Hence the finite sum telescopes and

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{p\to+\infty} m(E_p) .$$

We conclude that our formula in **(a)** is proved.

(b) Let $E_j = F_1 \setminus F_j$ for $j \in \mathbb{N}$. Thus $\{E_j\}$ is an increasing sequence of measurable sets. If we apply part **(a)** of the present theorem, then we may infer that

$$\begin{aligned} m\left(\bigcup_{j=1}^{\infty} E_j\right) &= \lim_{j\to\infty} m(E_j) \\ &= \lim_{n\to\infty} [m(F_1) - m(F_n)] \\ &= m(F_1) - \lim_{n\to\infty} m(F_n)\,. \end{aligned}$$

Since $\cup_{j=1}^{\infty} E_j = F_1 \setminus \cap_{n=1}^{\infty} F_n$, we may conclude from the theorem that

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = m(F_1) - m\left(\bigcap_{n=1}^{\infty} F_n\right)\,.$$

We conclude this discussion by combining the last two equations to obtain the desired equality. □