Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Signed Measures and the Hahn Decomposition

Definition: Let $\mathcal{X}$ be a σ-algebra on a set X. A function $\lambda : \mathcal{X} \to \mathbb{R}$ is called a *signed measure* if

(i) $\lambda(\emptyset) = 0$;

(ii) If E_j are measurable and pairwise disjoint then

$$\lambda\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \lambda(E_j)\,.$$

We see that a signed measure is very much like a measure except that a measure is defined to take only positive values and 0 while a signed measure is allowed to also take negative values. Notices that we *do not* allow a signed measure to take values in the extended reals. We also remind the reader that condition **(ii)** is called countable additivity.

Definition: Let λ be a signed measure on the σ-algebra $\mathcal{X}$. A set $P \in \mathcal{X}$ is said to be *positive* with respect to λ if $\lambda(E \cap P) \geq 0$ for any $E \in \mathcal{X}$. A set $N \in \mathcal{X}$ is said to be *negative* with respect to λ if $\lambda(E \cap N) \leq 0$ for any $E \in \mathcal{X}$. A set $M \in \mathcal{X}$ is said to be a *null set* for λ if $\lambda(E \cap M) = 0$ for any set $E \in \mathcal{X}$.

Now we can prove an important decomposition theorem for signed measures.

Theorem (Hahn Decomposition Theorem): If λ is a signed measure on the σ-algebra $\mathcal{X}$ on the set X, then there exist sets P and N in $\mathcal{X}$ with $X = P \cup N$, $P \cap N = \emptyset$, and such that P is positive and N is negative with respect to λ.

Proof: The class $\mathcal{P}$ of all positive sets is not empty, because it must at least contain the empty set $\emptyset$. Let

$$\alpha = \sup\{\lambda(A) : A \in \mathcal{P}\}\,.$$

Let $\{A_j\}$ be a sequence in $\mathcal{P}$ such that $\lim_{j\to\infty} \lambda(A_j) = \alpha$, and write $P = \cup_{j=1}^{\infty} A_j$.

Since the union of two positive sets is positive, the sequence $\{A_j\}$ can be chosen to be monotone increasing. We take this to be so. Clearly the set P defined above is a positive set for λ because

$$\lambda(E \cap P) = \lambda\left(E \cap \bigcup_{j=1}^{\infty} A_j\right) = \lambda\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) = \lim_{j\to\infty} \lambda(E \cap A_j) \geq 0\,.$$

Furthermore, $\alpha = \lim_{j\to\infty} \lambda(A_j) = \lambda(P) < \infty$.

We next prove that the set $N \equiv X \setminus P$ is a negative set. If not, then there is a measurable subset E of N so that $\lambda(E) > 0$. The set E cannot be a positive set, for if it were then $P \cup E$ would be positive with $\lambda(P \cup E) > \alpha$, contradicting the maximality of α. Hence E must itself contain sets with negative signed measure. Let n_1 be the least positive integer so that E contains a set E_1 in such that $\lambda(E_1) \le -1/n_1$. Now

$$\lambda(E \setminus E_1) = \lambda(E) - \lambda(E_1) > \lambda(E) > 0 \, .$$

However $E \setminus E_1$ cannot be a positive set; if it were then $P_1 = P \cup (E \setminus E_1)$ would be a positive set with $\lambda(P_1) > \alpha$. Therefore $E \setminus E_1$ contains sets with negative signed measure.

Now let n_2 be the least positive integer so that $E \setminus E_1$ contains a set E_2 in such that $\lambda(E_2) \le -1/n_2$. As before, $E \setminus (E_1 \cup E_2)$ is not a positive set, and we next let n_3 be the least positive integer such that $E \setminus (E_1 \cup E_2)$ contains a set E_3 in such that $\lambda(E_3) \le -1/n_3$.

Repeating this argument, we obtain a pairwise disjoint sequence $\{E_j\}$ of measurable sets such that $\lambda(E_j) \leq -1/n_j$. Set $F = \cup_{j=1}^{\infty} E_j$ so that

$$\lambda(F) = \sum_{j=1}^{\infty} \lambda(E_j) \leq -\sum_{j=1}^{\infty} \frac{1}{n_j} \leq 0 \, .$$

This shows that $1/n_j \to 0$ (because the series converges).

If now G is a measurable subset of $E \setminus F$ and $\lambda(G) < 0$, then $\lambda(G) < -1/(n_j - 1)$ for sufficiently large j, contradicting the fact that n_j is the least positive integer such that $E \setminus (E_1 \cup \cdots \cup E_{j-1})$ contains a set with signed measure less than $-1/n_j$. As a result, every measurable subset G of $E \setminus F$ must have $\lambda(G) \geq 0$. Hence $E \setminus F$ is a positive set for λ. Since

$$\lambda(E \setminus F) = \lambda(E) - \lambda(F) > 0 \,,$$

we conclude that $P \cup (E \setminus F)$ is a positive set with signed measure exceeding α. Again, that is a contradiction.

It follows that the set $N = X \setminus P$ is a negative set for λ. Thus we have obtained the desired decomposition of X. $\square$

A pair of measurable sets P, N satisfying the conclusion of the Hahn decomposition theorem is said to be a *Hahn decomposition* of X with respect to λ. In general the Hahn decomposition is not unique. In applications this lack of uniqueness is not important.

Lemma: If P_1, N_1 and P_2, N_2 are Hahn decompositions for X with respect to λ, and if $E \in \mathcal{X}$, then

$$\lambda(E \cap P_1) = \lambda(E \cap P_2) \quad \text{and} \quad \lambda(E \cap N_1) = \lambda(E \cap N_2)\,.$$

Proof: Since $E \cap (P_1 \setminus P_2)$ is contained in the positive set P_1 and also in the negative set N_2, we see that

$$\lambda(E \cap (P_1 \setminus P_2)) = 0 .$$

Now the lefthand side is $\lambda(E \cap P_1) - \lambda(E \cap (P_1 \cap P_2))$. Hence

$$\lambda(E \cap P_1) = \lambda(E \cap P_1 \cap P_2) .$$

Similarly, one can show that

$$\lambda(E \cap P_2) = \lambda(E \cap P_1 \cap P_2) .$$

In conclusion,

$$\lambda(E \cap P_1) = \lambda(E \cap P_2)\,.$$

The result for N_1 and N_2 follows immediately. $\square$

Definition: Let λ be a signed measure on $\mathcal{X}$ and let P, N be a Hahn decomposition for λ. The *positive variation* for λ is the finite measure

$$\lambda^+(E) \equiv \lambda(E \cap P) .$$

Likewise the *negative variation* for λ is the finite measure

$$\lambda^-(E) \equiv -\lambda(E \cap N) .$$

The *total variation* of λ is the measure $|\lambda|$ which is defined for $E \in \mathcal{X}$ by

$$|\lambda|(E) = \lambda^+(E) + \lambda^-(E) .$$

Note that we already encountered the ideas of positive and negative variation in an earlier lecture.

Remark: It is a consequence of the lemma above that the positive and negative variations of λ are well defined and in fact do not depend on the Hahn decomposition.

It is also worth noting that

$$\lambda(E) = \lambda(E \cap P) + \lambda(E \cap N) = \lambda^+(E) - \lambda^-(E) .$$

We now formalize the comments in this last remark.

Theorem (Jordan Decomposition Theorem) If λ is a signed measure on $\mathcal{X}$, then it is the difference of two finite measures on $\mathcal{X}$. That is to say, λ is the difference of λ^+ and λ^-. Furthermore, if $\lambda = \mu - \nu$ with μ, ν finite measures on $\mathcal{X}$, then

$$\mu(E) \geq \lambda^+(E) \quad \text{and} \quad \nu(E) \geq \lambda^-(E)$$

for all $E \in \mathcal{X}$.

Proof: We have already proved that $\lambda = \lambda^+ - \lambda^-$. Since μ and ν have nonnegative values, we have

$$\begin{aligned} \lambda^+(E) &= \lambda(E \cap P) \\ &= \mu(E \cap P) - \nu(E \cap P) \\ &\leq \mu(E \cap P) \\ &\leq \mu(E) . \end{aligned}$$

One shows similarly that $\lambda^-(E) \leq \nu(E)$. □