Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Now we identify the positive and negative variations of a signed measure λ in a fashion similar to what we saw in the earlier lemma.

Theorem: If f is integrable with respect to the measure space $(X, \mathcal{X}, \mu)$, and if λ is defined by

$$\lambda(E) = \int_E f \, d\mu \, ,$$

then λ^+, λ^-, and $|\lambda|$ are given for $E \in \mathcal{X}$ by

$$\lambda^+(E) = \int_E f^+ \, d\mu \quad , \quad \lambda^-(E) = \int_E f^- \, d\mu \, ,$$

$$|\lambda|(E) = \int_E |f| \, d\mu \, .$$

Proof: Let $P_f = \{x \in X : f(x) \geq 0\}$ and $N_f = \{x \in X : f(x) \leq 0\}$. Then $X = P_f \cup N_f$ and $P_f \cap N_f$ is a null set for λ. If $E \in \mathcal{X}$, then clearly $\lambda(E \cap P_f) \geq 0$ and $\lambda(E \cap N_f) \leq 0$. Hence P_f, N_f is a Hahn decomposition for λ. The result now follows. $\square$

The Radon-Nikodým Theorem

We begin with some terminology.

Definition: Let λ, μ be measures on a σ-algebra $\mathcal{X}$. We say that λ is *absolutely continuous* with respect to μ if, whenever $E \in \mathcal{X}$ and $\mu(E) = 0$, then $\lambda(E) = 0$. We then write $\lambda \ll \mu$.

Lemma: Let $\mathcal{X}$ be a σ-algebra and let λ and μ be finite measures on $\mathcal{X}$. Then $\lambda << \mu$ if and only if, for each $\epsilon > 0$, there exists a $\delta > 0$ so that $E \in \mathcal{X}$ and $\mu(E) < \delta$ both imply that $\lambda(E) < \epsilon$.

Proof: Suppose that the condition at the end of the second sentence is satisfied. If $\mu(F) = 0$, then let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\lambda(E) < \epsilon$. So certainly $\lambda(F) < \epsilon$. Since this is true for every $\epsilon > 0$, we see that $\lambda(F) = 0$. Hence $\lambda \ll \mu$.

Conversely, suppose that there is an $\epsilon > 0$ and sets $E_j \in \mathcal{X}$ so that $\mu(E_j) < 2^{-j}$ and $\lambda(E_j) \geq \epsilon$. Let $F_j = \cup_{k=j}^{\infty} E_k$ so that $\mu(F_j) < 2^{-j+1}$ and $\lambda(F_j) \geq \epsilon$. Since $\{F_j\}$ is a decreasing sequence of measurable sets, we see that

$$\mu\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j) = 0$$

and

$$\lambda\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \lambda(F_j) \geq \epsilon\,.$$

We see then that λ is *not* absolutely continuous with respect to μ. Contradiction. □

Theorem (Radon-Nikodým) Let λ and μ be σ-finite measures defined on a σ-algebra $\mathcal{X}$. Suppose that λ is absolutely continuous with respect to μ. Then there is a measurable function f on $\mathcal{X}$ such that

$$\lambda(E) = \int_E f \, d\mu \quad \text{for all} \quad E \in \mathcal{X} \, .$$

The function f is uniquely determined almost everywhere. We denote f by $d\lambda/d\mu$ and call it the *Radon-Nikodým derivative of* λ *with respect to* μ.

Proof: We shall first prove the result under the additional hypothesis that μ and λ are finite measures.

If $c > 0$, then let $P(c)$, $N(c)$ be a Hahn decomposition of X for the signed measure $\lambda - c\mu$. If $j \in \mathbb{N}$, then consider the measurable sets

$$A_1 = N(c) \ , \quad A_{j+1} = N\big((j+1)c\big) \setminus \bigcup_{\ell=1}^{j} A_\ell \, .$$

Clearly the sets A_j, $j \in \mathbb{N}$, are pairwise disjoint and

$$\bigcup_{j=1}^{k} N(jc) = \bigcup_{j=1}^{k} A_j \, .$$

It follows that

$$A_j = N(jc) \setminus \bigcup_{\ell=1}^{j-1} N(\ell c) = N(jc) \cap \bigcap_{\ell=1}^{j-1} P(\ell c) \,.$$

As a result, if E is a measurable subset of A_j, then $E \subseteq N(jc)$ and $E \subseteq P((j-1)c)$ so that

$$(j-1)c\mu(E) \leq \lambda(E) \leq jc\mu(E) \,.$$

Define B by

$$B = X \setminus \bigcup_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} P(jc)\,.$$

Thus $B \subseteq P(jc)$ for all $j \in \mathbb{N}$. This tells us that

$$0 \leq jc\mu(B) \leq \lambda(B) \leq \lambda(X) < +\infty$$

for all $j \in \mathbb{N}$, so that $\mu(B) = 0$. Since $\lambda \mu$, we see that $\lambda(B) = 0$.

Now let f_c be defined by

$$f_c(x) = \begin{cases} (j-1)c & \text{if} \quad x \in A_j \\ 0 & \text{if} \quad x \in B\,. \end{cases}$$

If E is any measurable set, then E is the union of the pairwise disjoint sets $E \cap B$ and $E \cap A_j$ for $j \in \mathbb{N}$. Hence it follows from the above that

$$\int_E f_c \, d\mu \leq \lambda(E) \leq \int_E (f_c + c) \, d\mu \leq \int_E f_c \, d\mu + c\mu(X) \, .$$

We next employ the preceding construction with $c = 2^{-j}$, $j \in \mathbb{N}$, to obtain a sequence of functions denoted by f_j. Thus we have

$$\int_E f_j \, d\mu \le \lambda(E) \le \int_E f_j \, d\mu + 2^{-j}\mu(X) \, ,$$

for all $j \in \mathbb{N}$. Let $k \ge j$ and note that

$$\int_E f_j \, d\mu \le \lambda(E) \le \int_E f_k \, d\mu + 2^{-k}\mu(X) \, ,$$

$$\int_E f_k \, d\mu \le \lambda(E) \le \int_E f_j \, d\mu + 2^{-j}\mu(X) \, .$$

From this we see that

$$\left| \int_E (f_j - f_k) \, d\mu \right| \le 2^{-j+1}\mu(X)$$

for all $E \in \mathcal{X}$.

If we let E range over the sets where the integrand is positive or negative and combine all these, we conclude that

$$\int |f_j - f_k| \, d\mu \leq 2^{-j+2} \mu(X)$$

whenever $k \geq j$. Thus $\{f_j\}$ converges in mean to a function f. Since the f_j are nonnegative, we may conclude that f is nonnegative.

Furthermore,

$$\left| \int_E f_j \, d\mu - \int_E f \, d\mu \right| \leq \int_E |f_j - f| \, d\mu \leq \int |f_j - f| \, d\mu \, .$$

Hence we may conclude from the above that

$$\lambda(E) = \lim_{j \to \infty} \int_E f_j \, d\mu = \int_E f \, d\mu$$

for all $E \in$. This completes the proof of the existence assertion of the theorem in the special case when λ and μ are finite measures.

For the uniqueness, notice that if

$$\lambda(E) = \int_E f_1 \, d\mu = \int_E f_2 \, d\mu \,,$$

then

$$\int_E (f_1 - f_2) \, d\mu = 0$$

for every $E \in \mathcal{X}$. From this it follows that $f_1 \equiv f_2$ almost everywhere.

We omit the proof of the more general case, but refer the reader to the book of Bartle for the details. Likewise for the uniqueness part of the theorem. □

The function f whose existence is established in the theorem is usually called the *Radon-Nikodým derivative* of λ with respect to μ. It is denoted by $d\lambda/d\mu$. The function f is not necessarily integrable unless λ is finite.

We have seen that a measure λ is absolutely continuous with respect to a measure μ when sets which have small μ measure also have small λ measure. The dialectic opposite of "absolutely continuous" is "singular."

Definition: Two measures λ and μ on a σ-algebra $\mathcal{X}$ on a set X are said to be *mutually singular* if there are disjoint sets $A, B \in \mathcal{X}$ so that $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. In these circumstances we write $\lambda \perp \mu$.

Although this definition is plainly symmetric in λ and μ, it is still common to say that "λ is singular with respect to μ."

Theorem (Lebesgue Decomposition Theorem) Let λ and μ be σ-finite measures defined on a σ-algebra . There exists a measure λ_1 which is singular with respect to μ and another measure λ_2 which is absolutely continuous with respect to μ so that $\lambda = \lambda_1 + \lambda_2$. The measures λ_1 and λ_2 are unique.

Proof: Let $\nu = \lambda + \mu$. Thus ν is a σ-finite measure. Since λ and μ are both absolutely continuous with respect to ν, the Radon-Nikodým theorem tells us that there are nonegative, measurable functions f, g such that

$$\lambda(E) = \int_E f \, d\nu \quad , \quad \mu(E) = \int_E g \, d\nu$$

for all $E \in$. Let $A = \{x : g(x) = 0\}$ and $B = \{x : g(x) > 0\}$. We see immediately that $A \cap B = \emptyset$ and $X = A \cup B$. This A and B are the two sets that we seek.

Define, for $E \in$,

$$\lambda_1(E) = \lambda(E \cap A) \quad \text{and} \quad \lambda_2(E) = \lambda(E \cap B)\,.$$

Because $\mu(A) = 0$, we see that $\lambda_1 \perp \mu$. To understand that $\lambda_2 << \mu$, notice that if $\mu(E) = 0$, then

$$\int_E g \, d\nu = 0\,.$$

Thus $g(x) = 0$ for ν-almost all $x \in E$. Hence $\nu(E \cap B) = 0$. Since $\lambda << \nu$,

$$\lambda_2(E) = \lambda(E \cap B) = 0\,.$$

Clearly $\lambda = \lambda_1 + \lambda_2$, so we have the required decomposition.

For the uniqueness, notice that if α is a measure such that $\alpha << \mu$ and if $\alpha \perp \mu$ also, then it must be that $\alpha = 0$. See also Exercise 7 at the end of the chapter. $\square$