Figure: This is your instructor.
The Lebesgue Integral
In this lecture we develop some ideas from functional analysis. In particular, we shall prove representation theorems for bounded linear functionals on $L^p$.

**Definition:** A linear functional on $L^p$ is a mapping $\varphi : L^p \to \mathbb{R}$ which is linear.

The linear functional $\varphi$ is **bounded** if there is a constant $M > 0$ such that

$$|\varphi(f)| \leq M\|f\|_{L^p}$$

for all $f \in L^p$. In this case, the bound or norm of the functional is defined to be

$$\|\varphi\| = \sup\{|\varphi(f)| : f \in L^p, \|f\|_{L^p} \leq 1\}.$$
Example: Fix $1 \leq p < \infty$. Let $q = p/(p-1)$ and let $g \in L^q$. Define a linear functional $\varphi$ on $L^p$ by

$$\varphi(f) = \int fg \, d\mu.$$ 

Then $\varphi$ is a linear functional with norm at most $\|g\|_{L^q}$. Just use Hölder’s inequality to verify this assertion.

And in fact we leave it to the reader to check that the norm actually equals $\|g\|_{L^q}$. To see this, just assume $g \geq 0$ and let $f = g^{1/(p-1)}$. This will result in the equality

$$\varphi(f) = \|f\|_{L^p} \cdot \|g\|_{L^q}.$$
The Riesz theorem gives a converse to the result in the example. We begin with a lemma. Note that a linear functional $\varphi$ is called positive if $\varphi(f) \geq 0$ for all $f \in L^p$ such that $f \geq 0$.

**Lemma:** Let $\varphi$ be a bounded linear functional on $L^p$. Then there exist two positive bounded linear functionals $\varphi^+$ and $\varphi^-$ such that

$$\varphi(f) = \varphi^+(f) - \varphi^-(f)$$

for all $f \in L^p$. 
Proof: If $f \geq 0$, then define

$$\varphi^+(f) = \sup\{\varphi(g) : g \in L^p, 0 \leq g \leq f\}.$$ 

Clearly $\varphi^+(cf) = c\varphi^+(f)$ for $c \geq 0$ and $f \geq 0$.

If $0 \leq g_j \leq f_j$ for $j = 1, 2$, then

$$\varphi(g_1) + \varphi(g_2) = \varphi(g_1 + g_2) \leq \varphi^+(f_1 + f_2).$$

Taking suprema over all such $g_j \in L^p$, we find that

$$\varphi^+(f_1) + \varphi^+(f_2) \leq \varphi^+(f_1 + f_2).$$
Conversely, if $0 \leq h \leq f_1 + f_2$, we let $g_1 = \max(h - f_2, 0)$ and $g_2 = \min(h, f_2)$. We infer then that $g_1 + g_2 = h$ and also $0 \leq g_j \leq f_j$ for $j = 1, 2$. Thus

$$\varphi(h) = \varphi(g_1) + \varphi(g_2) \leq \varphi^+(f_1) + \varphi^+(f_2)$$

for all $f_j \in L^p$ with $f_j \geq 0$. Taking the supremum over $h$ then yields that

$$\varphi^+(f_1 + f_2) \leq \varphi^+(f_1) + \varphi^+(f_2).$$

Combining our last two displayed inequalities now shows that $\varphi^+$ is linear.
If $f \in L^p$ is arbitrary, then we define

$$\varphi^+(f) = \varphi^+(f^+) + \varphi^+(f^-),$$

where $f^+$ and $f^-$ are respectively the positive and negative parts of $f$.

It is easy to see that $\varphi^+$ is a bounded linear functional on $L^p$. Next, for $f \in L^p$, we define

$$\varphi^-(f) = \varphi^+(f) - \varphi(f).$$

We see that $\varphi^-$ is plainly a bounded linear functional (because $\varphi^+$ and $\varphi$ are). From the definition of $\varphi^+$ we see that $\varphi^-$ is also a positive linear functional. Finally, it is obvious that $\varphi = \varphi^+ - \varphi^-$. 

\[\square\]
Theorem (Riesz Representation Theorem, first version): If \((X, \mathcal{X}, \mu)\) is a \(\sigma\)-finite measure space and \(\varphi\) is a bounded linear functional on \(L^1\), then there exists a function \(g \in L^\infty\) such that the equation

\[
\varphi(f) = \int fg \, d\mu
\]

holds for all \(f \in L^1\). Furthermore, \(\|\varphi\| = \|g\|_{L^\infty}\). Also \(g \geq 0\) if \(\varphi\) is a positive linear functional.
Proof: We first treat the case that $\mu$ is a finite measure and $\varphi$ is positive. Define $\lambda : \mathcal{X} \rightarrow \mathbb{R}$ by

$$\lambda(E) = \varphi(\chi_E).$$

Clearly $\lambda(\emptyset) = 0$. If $\{E_j\}$ is an increasing sequence in $\mathcal{X}$ and $E = \bigcup_j E_j$, then $\{\chi_{E_j}\}$ converges pointwise to $\chi_E$. Since $\mu(X) < \infty$, it follows from the Lebesgue Monotone Convergence Theorem that the same sequence converges in $L^1$ to $\chi_E$.

Since

$$0 \leq \lambda(E) - \lambda(E_j) = \varphi(\chi_E) - \varphi(\chi_{E_j}) = \varphi(\chi_E - \chi_{E_j}) \leq \|\varphi\| \cdot \|\chi_E - \chi_{E_j}\|_{L^1},$$

we see (since $\lambda(E_j) \rightarrow \lambda(E)$) that $\lambda$ is a measure. Furthermore, if $F \in \mathcal{X}$ and $\mu(F) = 0$, then $\lambda(F) = 0$, hence $\lambda \ll \mu$. 
Applying the Radon-Nikodým theorem, we obtain a nonnegative, measurable function $g : \mathcal{X} \to \mathbb{R}$ such that

$$\varphi(\chi_E) = \lambda(E) = \int \chi_E \cdot g \, d\mu$$

for all $E \in \mathcal{X}$. Linearity now implies that

$$\varphi(h) = \int h \cdot g \, d\mu$$

for all $\mathcal{X}$-measurable, simple functions $h$.

If $f$ is a nonnegative function in $L^1$, we let $\{h_j\}$ be a monotone increasing sequence of simple functions converging almost everywhere and in $L^1$ to $f$. From the boundedness of $\varphi$, it is easily seen that $\varphi(f) = \lim_{j \to \infty} \varphi(h_j)$. Furthermore, the monotone convergence theorem tells us that

$$\varphi(f) = \lim_{j \to \infty} \int h_j g \, d\mu = \int fg \, d\mu.$$  

By linearity, this equality holds for arbitrary $f \in L^1$. That completes the case that $\mu$ is a finite measure.
We leave the case of $\sigma$-finite $\mu$ for the reader to consult in Bartle. It is an exercise to check that $\|\varphi\| = \|g\|_{L^\infty}$. 

What we have shown with this first version of the Riesz representation theorem is that the dual of the Banach space $L^1$ is $L^\infty$. We conclude this chapter by now proving a version of the theorem for $L^p$. ✷
Theorem (Riesz Representation Theorem, second version):
Let \((X, \mathcal{X}, \mu)\) be an arbitrary measure space. Let \(\varphi\) be a bounded linear functional on \(L^p\), \(1 \leq p < \infty\). Then there exists a \(g \in L^q\), \(q = p/(p - 1)\), so that
\[
\varphi(f) = \int fg \, d\mu
\]
holds for all \(f \in L^p\). Moreover, \(\|\varphi\| = \|g\|_{L^q}\).
Proof: In the case that $\mu$ is finite, the proof of the preceding version of the Riesz theorem requires only minor changes to show that there exists a $g \in L^q$ with $\|\varphi\| = \|g\|_{L^q}$ and such that

$$\varphi(f) = \int fg \, d\mu$$

for all $f \in L^p$. This result can be extended to the $\sigma$-finite case—see Bartle.

Now we complete the proof by noting that a bounded linear functional “vanishes off of a $\sigma$-finite set.” Indeed, let $\{f_j\}$ be a sequence in $L^p$ so that $\|f_j\| = 1$ and

$$\varphi(f_j) \geq \|\varphi\| \cdot \left(1 - \frac{1}{j}\right).$$

There exists a $\sigma$-finite set $X_0 \in \mathcal{X}$ outside of which all the $f_j$ vanish. Let $E \in \mathcal{X}$ with $E \cap X_0 = \emptyset$. Then

$$\|f_j \pm t \chi_E\|_{L^p} \leq (1 + t^p \mu(E))^{1/p}$$

for $t \geq 0$ when $\mu(E) < \infty$. 
Furthermore, since

$$
\varphi(f_j) + \varphi(\pm t\chi_E) \leq |\varphi(f_j \pm t\chi_E)|,
$$

we see that

$$
|\varphi(t\chi_E)| \leq \|\varphi\| \cdot \left\{ (1 + t^p \mu(E))^{1/p} - \left(1 - \frac{1}{j}\right) \right\}
$$

for all $j \in \mathbb{N}$.

First let $j \to \infty$ and then divide by $t > 0$ to obtain

$$
|\varphi(\chi_E)| \leq \|\varphi\| \cdot \frac{(1 + t^p \mu(E))^{1/p} - 1}{t}.
$$

If we apply L’Hôpital’s rule as $t \to 0^+$, we may conclude (as long as $p > 1$) that $\varphi(\chi_E) = 0$ for any $E \in \mathcal{X}$, $\mu(E) < \infty$, outside of the $\sigma$-finite set $X_0$. Thus, if $f$ is any function in $L^p$ such that $X_0 \cap \{x \in X : f(x) \neq 0\} = \emptyset$, then it follows that $\varphi(f) = 0$. 
Now we can apply the preceding argument to find a function $g$ on $X_0$ which represents $\varphi$ and we extend $g$ to all of $X$ by requiring that it vanish on the complement of $X_0$. This gives the desired function.

It should be noted that this last theorem is false for $p = \infty$. In fact the dual of $L^{\infty}$ is quite a complicated space that cannot be described explicitly.