Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

In this lecture we develop some ideas from functional analysis. In particular, we shall prove representation theorems for bounded linear functionals on L^p.

Definition: A *linear functional* on L^p is a mapping $\varphi : L^p \to \mathbb{R}$ which is linear.

The linear functional φ is *bounded* if there is a constant $M > 0$ such that

$$|\varphi(f)| \leq M\|f\|_{L^p}$$

for all $f \in L^p$. In this case, the *bound* or *norm* of the functional is defined to be

$$\|\varphi\| = \sup\{|\varphi(f)| : f \in L^p, \|f\|_{L^p} \leq 1\} .$$

Example: Fix $1 \leq p < \infty$. Let $q = p/(p-1)$ and let $g \in L^q$. Define a linear functional φ on L^p by

$$\varphi(f) = \int fg \, d\mu \, .$$

Then φ is a linear functional with norm at most $\|g\|_{L^q}$. Just use Hölder's inequality to verify this assertion.

And in fact we leave it to the reader to check that the norm actually equals $\|g\|_{L^q}$. To see this, just assume $g \geq 0$ and let $f = g^{1/(p-1)}$. This will result in the equality $\varphi(f) = \|f\|_{L^p} \cdot \|g\|_{L^q}$.

The Riesz theorem gives a converse to the result in the example. We begin with a lemma. Note that a linear functional φ is called *positive* if $\varphi(f) \geq 0$ for all $f \in L^p$ such that $f \geq 0$.

Lemma: *Let φ be a bounded linear functional on L^p. Then there exist two positive bounded linear functionals φ^+ and φ^- such that*

$$\varphi(f) = \varphi^+(f) - \varphi^-(f)$$

for all $f \in L^p$.

Proof: If $f \geq 0$, then define

$$\varphi^+(f) = \sup\{\varphi(g) : g \in L^p, 0 \leq g \leq f\} .$$

Clearly $\varphi^+(cf) = c\varphi^+(f)$ for $c \geq 0$ and $f \geq 0$.
If $0 \leq g_j \leq f_j$ for $j = 1, 2$, then

$$\varphi(g_1) + \varphi(g_2) = \varphi(g_1 + g_2) \leq \varphi^+(f_1 + f_2) .$$

Taking suprema over all such $g_j \in L^p$, we find that

$$\varphi^+(f_1) + \varphi^+(f_2) \leq \varphi^+(f_1 + f_2) .$$

Conversely, if $0 \leq h \leq f_1 + f_2$, we let $g_1 = \max(h - f_2, 0)$ and $g_2 = \min(h, f_2)$. We infer then that $g_1 + g_2 = h$ and also $0 \leq g_j \leq f_j$ for $j = 1, 2$. Thus

$$\varphi(h) = \varphi(g_1) + \varphi(g_2) \leq \varphi^+(f_1) + \varphi^+(f_2)$$

for all $f_j \in L^p$ with $f_j \geq 0$. Taking the supremum over h then yields that

$$\varphi^+(f_1 + f_2) \leq \varphi^+(f_1) + \varphi^+(f_2)\,.$$

Combining our last two displayed inequalities now shows that φ^+ is linear.

If $f \in L^p$ is arbitrary, then we define

$$\varphi^+(f) = \varphi^+(f^+) + \varphi^+(f^-) ,$$

where f^+ and f^- are respectively the positive and negative parts of f.

It is easy to see that φ^+ is a bounded linear functional on L^p. Next, for $f \in L^p$, we define

$$\varphi^-(f) = \varphi^+(f) - \varphi(f) .$$

We see that φ^- is plainly a bounded linear functional (because φ^+ and φ are). From the definition of φ^+ we see that φ^- is also a positive linear functional. Finally, it is obvious that
$\varphi = \varphi^+ - \varphi^-$. □

Theorem (Riesz Representation Theorem, first version): *If $(X, \mathcal{X}, \mu)$ is a σ-finite measure space and φ is a bounded linear functional on L^1, then there exists a function $g \in L^\infty$ such that the equation*

$$\varphi(f) = \int fg \, d\mu$$

holds for all $f \in L^1$. Furthermore, $\|\varphi\| = \|g\|_{L^\infty}$. Also $g \geq 0$ if φ is a positive linear functional.

Proof: We first treat the case that μ is a finite measure and φ is positive. Define $\lambda : \mathcal{X} \to \mathbb{R}$ by

$$\lambda(E) = \varphi(\chi_E) \,.$$

Clearly $\lambda(\emptyset) = 0$. If $\{E_j\}$ is an increasing sequence in $\mathcal{X}$ and $E = \cup_j E_j$, then $\{\chi_{E_j}\}$ converges pointwise to χ_E. Since $\mu(X) < \infty$, it follows from the Lebesgue Monotone Convergence Theorem that the same sequence converges in L^1 to χ_E.

Since

$$\begin{aligned} 0 &\leq \lambda(E) - \lambda(E_j) \\ &= \varphi(\chi_E) - \varphi(\chi_{E_j}) \\ &= \varphi(\chi_E - \chi_{E_j}) \\ &\leq \|\varphi\| \cdot \|\chi_E - \chi_{E_j}\|_{L^1} \,, \end{aligned}$$

we see (since $\lambda(E_j) \to \lambda(E)$) that λ is a measure. Furthermore, if $F \in \mathcal{X}$ and $\mu(F) = 0$, then $\lambda(F) = 0$, hence $\lambda << \mu$.

Applying the Radon-Nikodým theorem, we obtain a nonnegative, measurable function $g : X \to \mathbb{R}$ such that

$$\varphi(\chi_E) = \lambda(E) = \int \chi_E \cdot g \, d\mu$$

for all $E \in \mathcal{X}$. Linearity now implies that

$$\varphi(h) = \int h \cdot g \, d\mu$$

for all $\mathcal{X}$-measurable, simple functions h.

If f is a nonnegative function in L^1, we let $\{h_j\}$ be a monotone increasing sequence of simple functions converging almost everywhere and in L^1 to f. From the boundedness of φ, it is easily seen that $\varphi(f) = \lim_{j\to\infty} \varphi(h_j)$. Furthermore, the monotone convergence theorem tells us that

$$\varphi(f) = \lim_{j\to\infty} \int h_j g \, d\mu = \int fg \, d\mu \, .$$

By linearity, this equality holds for arbitrary $f \in L^1$. That completes the case that μ is a finite measure.

We leave the case of σ-finite μ for the reader to consult in Bartle. It is an exercise to check that $\|\varphi\| = \|g\|_{L^\infty}$. $\square$

What we have shown with this first version of the Riesz representation theorem is that the dual of the Banach space L^1 is L^∞. We conclude this chapter by now proving a version of the theorem for L^p.

Theorem (Riesz Representation Theorem, second version): *Let $(X, \mathcal{X}, \mu)$ be an arbitrary measure space. Let φ be a bounded linear functional on L^p, $1 \le p < \infty$. Then there exists a $g \in L^q$, $q = p/(p-1)$, so that*

$$\varphi(f) = \int fg \, d\mu$$

holds for all $f \in L^p$. Moreover, $\|\varphi\| = \|g\|_{L^q}$.

Proof: In the case that μ is finite, the proof of the preceding version of the Riesz theorem requires only minor changes to show that there exists a $g \in L^q$ with $\|\varphi\| = \|g\|_{L^q}$ and such that

$$\varphi(f) = \int fg \, d\mu$$

for all $f \in L^p$. This result can be extended to the σ-finite case—see Bartle.

Now we complete the proof by noting that a bounded linear functional "vanishes off of a σ-finite set." Indeed, let $\{f_j\}$ be a sequence in L^p so that $\|f_j\| = 1$ and

$$\varphi(f_j) \geq \|\varphi\| \cdot \left(1 - \frac{1}{j}\right) .$$

There exists a σ-finite set $X_0 \in \mathcal{X}$ outside of which all the f_j vanish. Let $E \in \mathcal{X}$ with $E \cap X_0 = \emptyset$. Then

$$\|f_j \pm t\chi_E\|_{L^p} \leq (1 + t^p \mu(E))^{1/p}$$

for $t \geq 0$ when $\mu(E) < \infty$.

Furthermore, since

$$\varphi(f_j) + \varphi(\pm t\chi_E) \le |\varphi(f_j \pm t\chi_E)|,$$

we see that

$$|\varphi(t\chi_E)| \le \|\varphi\| \cdot \left\{(1 + t^p\mu(E))^{1/p} - \left(1 - \frac{1}{j}\right)\right\}$$

for all $j \in \mathbb{N}$.

First let $j \to \infty$ and then divide by $t > 0$ to obtain

$$|\varphi(\chi_E)| \le \|\varphi\| \cdot \frac{(1 + t^p\mu(E))^{1/p} - 1}{t}.$$

If we apply L'Hôspital's rule as $t \to 0^+$, we may conclude (as long as $p > 1$) that $\varphi(\chi_E) = 0$ for any $E \in \mathcal{X}$, $\mu(E) < \infty$, outside of the σ-finite set X_0. Thus, if f is any function in L^p such that $X_0 \cap \{x \in X : f(x) \neq 0\} = \emptyset$, then it follows that $\varphi(f) = 0$.

Now we can apply the preceding argument to find a function g on X_0 which represents φ and we extend g to all of X by requiring that it vanish on the complement of X_0. This gives the desired function. □

It should be noted that this last theorem is false for $p = \infty$. In fact the dual of L^∞ is quite a complicated space that cannot be described explicitly.