Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Outer Measure

In the preceding lectures we have presented some quite elementary methods for creating measures. Now we introduce some more sophisticated techniques. And we also literally *construct* Lebesgue measure from the operation of measuring the lengths of intervals.

In fact we will be taking a new approach to the idea of measure. We begin by defining an algebra, which is something a bit weaker than a σ-algebra. Namely, a σ-algebra respects countable unions while an algebra only respects finite unions.

The good thing about an algebra is that it is very easy to specify a measure on an algebra. Then we explain how one can construct a σ-algebra that contains the algebra; finally we extend the measure to that σ-algebra. This is in fact how we will explicitly construct Lebesgue measure this time around.

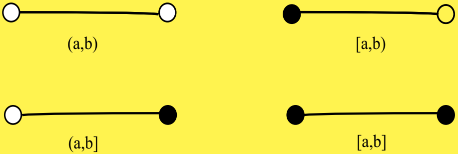
Definition: We define the *length* of a bounded interval of the form

$$(a, b) \quad \text{or} \quad [a, b) \quad \text{or} \quad (a, b] \quad \text{or} \quad [a, b]$$

to be $b - a$. The length of an interval of the form

$$(-\infty, b] \quad \text{or} \quad (-\infty, b) \quad \text{or} \quad (b, \infty) \quad \text{or} \quad [b, \infty)$$

is the extended real number $+\infty$. We denote the length of a set S by $\ell(S)$. See the next two figures.


(a,b)
[a,b)
(a,b]
[a,b]

Definition: If we have finitely many pairwise disjoint intervals of the above form, then the length of their union is defined to be the sum of the lengths of the component intervals. This aggregate length could be a finite, nonnegative number or it could be $+\infty$.

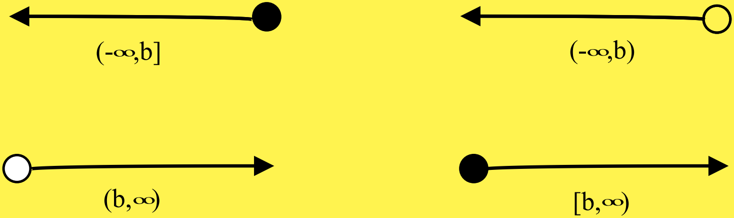


Figure 9.2: Intervals of infinite length.

Now we define the notion of an algebra.

Definition: Let X be a given set. A family $\mathcal{A}$ of subsets of X is called an *algebra* or a *field* if

(i) $\emptyset$, X both belong to $\mathcal{A}$;

(ii) If E belongs to $\mathcal{A}$, then its complement $X \setminus E$ also belongs to $\mathcal{A}$;

(iii) If $E_1, E_2, \ldots, E_k$ belong to $\mathcal{A}$, then also their union $\cup_{j=1}^{k} E_j$ belongs to $\mathcal{A}$.

Now we do something a bit unusual. We define a measure on an algebra to be a scalar-valued function that respects countable unions.

Definition: Let $\mathcal{A}$ be an algebra of subsets of a set X. A *measure* on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \widehat{\mathbb{R}}^+$ satisfying:

(a) $\mu(\emptyset) = 0$;

(b) $\mu(E) \geq 0$ for all $E \in \mathcal{A}$;

(c) If $\{E_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{A}$ such that $\cup_{j=1}^{\infty} E_j$ also belongs to $\mathcal{A}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) \, .$$

Notice, in this definition, that we had to hypothesize that the union of the E_j lies in $\mathcal{A}$. That does *not* follow automatically from the definition of an algebra.

Lemma: *The collection $\mathcal{F}$ of all finite unions of intervals as described above is an algebra of subsets of $\mathbb{R}$ and length is a measure on $\mathcal{F}$.*

Proof: It is apparent that $\mathcal{F}$ is an algebra. If we use ℓ to denote the length function, then part **(a)** and part **(b)** of the definition are obvious. To prove part **(c)**, it suffices to show that if one of the sets of the form discussed above is the union of a countable, pairwise disjoint collection of sets of this form, then the lengths add up correctly. We shall just treat an interval of the form $(a, b]$, leaving the other possibilities as an exercise for the reader.

Figure: The union of half-open intervals.

Suppose now that

$$(a, b] = \bigcup_{j=1}^{\infty} (a_j, b_j] \, ,$$

where the intervals $(a_j, b_j]$ are pairwise disjoint. Refer to the last figure. Let $(a_1, b_1]$, $(a_2, b_2]$, ..., $(a_k, b_k]$ be any finite collection of such intervals, and suppose that

$$a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_{k-1} < b_{k-1} \le a_k < b_k \le b \, .$$

[We may need to renumber indices to achieve this goal, but that is simply a formality.] Now we have

$$\begin{aligned} \sum_{j=1}^{k} \ell((a_j, b_j]) &= \sum_{j=1}^{k} (b_j - a_j) \\ &\le b_k - a_1 \\ &\le b - a \\ &= \ell((a, b]) \, . \end{aligned}$$

Since the index k is arbitrary, we conclude that

$$\sum_{j=1}^{\infty} \ell((a_j, b_j]) \le \ell((a, b]) .$$

Conversely, let $\epsilon > 0$ and let $\{\epsilon_j\}$ be a sequence of positive numbers with $\sum_j \epsilon_j < \epsilon/2$. Consider now the intervals

$$I_j = (a_j - \epsilon_j, b_j + \epsilon_j) \ , \quad j \in \mathbb{N} .$$

We see that the open sets $\{I_j : j \in \mathbb{N}\}$ form a covering of the compact interval $[a, b]$. Hence there is a finite subcovering $I_1, I_2, \ldots, I_m$. By renumbering and discarding some possibly extra intervals, we may assume that

$$a_1 - \epsilon_1 < a \quad , \quad b < b_m + \epsilon_m \ ,$$

$$a_j - \epsilon_j < b_{j-1} + \epsilon_{j-1} \ , \quad j = 2, \ldots, m .$$

It follows from these inequalities that

$$\begin{aligned} b - a &\leq (b_m + \epsilon_m) - (a_1 - \epsilon_1) \\ &\leq \sum_{j=1}^{m} [(b_j + \epsilon_j) - (a_j - \epsilon_j)] \\ &\leq \sum_{j=1}^{m} (b_j - a_j) + \epsilon \\ &\leq \sum_{j=1}^{\infty} (b_j - a_j) + \epsilon\,. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\ell((a,b]) \leq \sum_{j=1}^{\infty} \ell((a_j, b_j])$. Combining this inequality with our previous inequality, we may conclude that the length function ℓ is countably additive on $\mathcal{F}$. □