Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Outer Measure

The next step in our program is to show that, if $\mathcal{A}$ is an algebra of subsets of a set X and if μ is a measure defined on $\mathcal{A}$, then there exists a σ-algebra $\widehat{\mathcal{A}}$ containing $\mathcal{A}$ and a measure $\widehat{\mu}$ defined on $\widehat{\mathcal{A}}$ so that $\widehat{\mu}(E) = \mu(E)$ for $E \in \mathcal{A}$. What we are saying then is that the measure μ can be extended from the algebra $\mathcal{A}$ to a measure on a σ-algebra $\widehat{\mathcal{A}}$ which contains $\mathcal{A}$.

Definition: Let $\mathcal{A}$ be an algebra of sets on X. Let μ be a measure on $\mathcal{A}$. If F is an arbitrary subset of X, then we define

$$\widehat{\mu}(F) = \inf \sum_{j=1}^{\infty} \mu(E_j)\,,$$

where the infimum is extended over all sequences $\{E_j\}$ of sets in $\mathcal{A}$ such that

$$F \subseteq \bigcup_{j=1}^{\infty} E_j\,.$$

The function $\widehat{\mu}$ that we have just defined is called the *outer measure* generated by μ. This terminology is a bit confusing, just because $\widehat{\mu}$ is usually *not* a measure. It does, however, have some of the properties of a measure.

Lemma: *The function $\widehat{\mu}$ of the above definition has the following properties:*

(a) $\widehat{u}(\emptyset) = 0$;

(b) $\widehat{u}(F) \geq 0$ *for* $F \subseteq X$;

(c) *If* $F \subseteq G$, *then* $\widehat{u}(F) \leq \widehat{u}(G)$;

(d) *If* $F \in \mathcal{A}$, *then* $\widehat{u}(F) = \mu(F)$;

(e) *If* $\{F_j\}$ *is a sequence of subsets of* X, *then*

$$\widehat{u}\left(\bigcup_{j=1}^{\infty} F_j\right) \leq \sum_{j=1}^{\infty} \widehat{u}(F_j) .$$

Proof: Statements **(a)**, **(b)**, and **(c)** are immediate consequences of the definition.

For **(d)**, note that since $\{F, \emptyset, \emptyset, \dots\}$ is a countable collection of sets in $\mathcal{A}$ whose union contain F, we see that

$$\widehat{\mu}(F) \leq \mu(F) + 0 + 0 + \cdots = \mu(F) .$$

Conversely, if $\{E_j\}$ is any sequence of elements of $\mathcal{A}$ with $F \subseteq \cup_j E_j$, then $F = \cup_j (F \cap E_j)$. Since μ is assumed to be a measure on $\mathcal{A}$, we see that

$$\mu(F) \leq \sum_{j=1}^{\infty} \mu(F \cap E_j) \leq \sum_{j=1}^{\infty} \mu(E_j) ,$$

from which we infer that $\mu(F) \leq \widehat{\mu}(F)$.

To prove **(e)**, let $\epsilon > 0$ and for each j choose a sequence $\{E_{jk}\}$ of sets in $\mathcal{A}$ so that

$$F_j \subseteq \bigcup_{k=1}^{\infty} E_{jk} \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(E_{jk}) \leq \widehat{\mu}(F_j) + \frac{\epsilon}{2^j} \, .$$

Since $\{E_{jk} : j, k \in \mathbb{N}\}$ is a countable collection of elements of $\mathcal{A}$ whose union contains $\cup F_j$, it follows from the definition of $\widehat{\mu}$ that

$$\widehat{\mu}\left(\bigcup_{j=1}^{\infty} F_j\right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(E_{jk}) \leq \sum_{j=1}^{\infty} \widehat{\mu}(F_j) + \epsilon \, .$$

Since ϵ is arbitrary, we have obtained the desired inequality. $\square$

Property **(e)** of the lemma is usually described by the phrase "$\widehat{\mu}$ is countably subadditive."

Remark: An important mathematical point must be made now. The outer measure $\widehat{\mu}$ assigns a length or measure to *every* subset of X. But of course we know from our considerations in Chapter 1 that that is not possible for a measure. So what is going on here?

The answer is that $\widehat{\mu}$ is *not* countably additive. In fact it is not necessarily finitely additive. What we shall do in our work below is to restrict $\widehat{u}$ to a smaller σ-algebra on which the outer measure *is* countably additive. The key result here is due to Constantin Carathéodory.

Definition: A subset E of X is said to be $\widehat{\mu}$-*measurable* if

$$\widehat{\mu}(A) = \widehat{\mu}(A \cap E) + \widehat{\mu}(A \setminus E)$$

for all subsets $A \subseteq X$. The collection of all $\widehat{\mu}$-measurable sets is denoted by $\widehat{\mathcal{A}}$.

The above condition defines an additivity property on $\widehat{\mu}$. Roughly speaking, a set E is $\widehat{\mu}$-measurable in case it and its complement are separated enough so that they divide an arbitrary set A in an additive fashion.

Theorem (Carathéodory Extension Theorem): The collection $\widehat{\mathcal{A}}$ of all $\widehat{\mu}$-measurable sets is a σ-algebra containing $\mathcal{A}$. Moreover, if $\{E_j\}$ is a pairwise disjoint sequence in $\widehat{\mathcal{A}}$, then

$$\widehat{\mu}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \widehat{\mu}(E_j) \, .$$

Proof: Plainly $\emptyset$ and X are $\widehat{\mu}$-measurable. Also, if $E \in \widehat{\mathcal{A}}$, then the complement $X \setminus E$ is also in $\widehat{\mathcal{A}}$.

For our first step of the proof, we shall show that $\widehat{\mathcal{A}}$ is closed under intersection. Suppose that E, F are $\widehat{\mu}$-measurable. Then, for any $A \subseteq X$,

$$\widehat{\mu}(A \cap F) = \widehat{\mu}(A \cap F \cap E) + \widehat{\mu}((A \cap F) \setminus E) .$$

Since $F \in \widehat{\mathcal{A}}$, we have

$$\widehat{\mu}(A) = \widehat{\mu}(A \cap F) + \widehat{\mu}(A \setminus F) .$$

Let $B = A \setminus (E \cap F)$. Then clearly $B \cap F = (A \cap F) \setminus E$ and $B \setminus F = A \setminus F$. Since $F \in \widehat{\mathcal{A}}$, we see that

$$\widehat{\mu}(A \setminus (E \cap F)) = \widehat{\mu}((A \cap F) \setminus E) + \widehat{\mu}(A \setminus F) .$$

Combining the last three equalities gives

$$\widehat{\mu}(A) = \widehat{\mu}(A \cap (E \cap F)) + \widehat{\mu}(A \setminus (E \cap F)) .$$

This proves that $E \cap F \in \widehat{\mathcal{A}}$. Since $\widehat{\mathcal{A}}$ is closed under intersection and complementation, we may conclude that $\widehat{\mathcal{A}}$ is an algebra.

Now assume that $E, F \in \widehat{\mathcal{A}}$ and that $E \cap F = \emptyset$. If we take A to be instead $A \cap (E \cup F)$ in the line above, we obtain

$$\widehat{\mu}(A \cap (E \cup F)) = \widehat{\mu}(A \cap E) + \widehat{\mu}(A \cap F) .$$

Letting now $A = X$, we see that $\widehat{\mu}$ is additive on $\widehat{\mathcal{A}}$.

Now we show that $\widehat{\mathcal{A}}$ is a σ-algebra and that $\widehat{\mu}$ is countably additive on $\widehat{\mathcal{A}}$. Let $\{E_j\}_{j=1}^{\infty}$ be a pairwise disjoint collection in $\widehat{\mathcal{A}}$. Set $E = \cup_j E_j$. From the previous paragraph we know that $F_k = \cup_{j=1}^{k} E_j$ belongs to $\widehat{\mathcal{A}}$.

Also, if A is any subset of X, then

$$\widehat{\mu}(A) = \widehat{\mu}(A \cap F_k) + \widehat{\mu}(A \setminus F_k) = \left[\sum_{j=1}^{k} \widehat{\mu}(A \cap E_j)\right] + \widehat{\mu}(A \setminus F_k) .$$

Since $F_k \subseteq E$, we may see that $A \setminus E \subseteq A \setminus F_k$. Letting $k \to \infty$ we may conclude from the above that

$$\sum_{j=1}^{\infty} \widehat{\mu}(A \cap E_j) + \widehat{\mu}(A \setminus E) \leq \widehat{\mu}(A) \, .$$

It follows now from the lemma above that

$$\widehat{\mu}(A \cap E) \leq \sum_{j=1}^{\infty} \widehat{\mu}(A \cap E_j)$$

and

$$\widehat{\mu}(A) \leq \widehat{\mu}(A \cap E) + \widehat{\mu}(A \setminus E) \, .$$

Combining the last three inequalities yields

$$\widehat{\mu}(A) = \widehat{\mu}(A \cap E) + \widehat{\mu}(A \setminus E) = \sum_{j=1}^{\infty} \widehat{\mu}(A \cap E_j) + \widehat{\mu}(A \setminus E) \, .$$

This shows in particular that $E = \cup_{j=1}^{\infty} E_j$ is $\widehat{\mu}$-measurable. Taking $A = E$, we obtain the desired equation.

The last thing that we must do is to show that $\mathcal{A} \subseteq \widehat{\mathcal{A}}$. It was proved in part **(d)** of the lemma that, if $E \in \mathcal{A}$, then $\widehat{\mu}(E) = \mu(E)$. It remains to show that E is $\widehat{\mu}$-measurable. So let A be an arbitrary subset of X. Using part **(e)** of the lemma, we see that

$$\widehat{\mu}(A) \leq \widehat{\mu}(A \cap E) + \widehat{\mu}(A \setminus E) \, .$$

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To prove the opposite inequality, let $\epsilon > 0$ and let $\{F_j\}$ be a sequence in $\mathcal{A}$ such that $A \subseteq \cup_j F_j$ and

$$\sum_{j=1}^{\infty} \mu(F_j) \leq \widehat{\mu}(A) + \epsilon\,.$$

Since $A \cap E \subseteq \cup_j(F_j \cap E)$ and $A \setminus E \subseteq \cup_j(F_j \setminus E)$, we may infer from part **(e)** of the lemma that

$$\widehat{\mu}(A \cap E) \leq \sum_{j=1}^{\infty} \mu(F_j \cap E) \quad \text{and} \quad \widehat{\mu}(A \setminus E) \leq \sum_{j=1}^{\infty} \mu(F_j \setminus E)\,.$$

Thus we may conclude that

$$\begin{aligned} \widehat{\mu}(A \cap E) + \widehat{\mu}(A \setminus E) &\leq \sum_{j=1}^{\infty} \Big\{ \mu(F_j \cap E) + \mu(F_j \setminus E) \Big\} \\ &= \sum_{j=1}^{\infty} \mu(F_j) \end{aligned}$$

Since ϵ was arbitrary, we have proved the desired inequality and the set E belongs to $\widehat{\mathcal{A}}$. □

Remark: The Carathéodory extension theorem shows that a measure μ on an algebra $\mathcal{A}$ can always be extended to a measure $\widehat{\mu}$ on a σ-algebra $\widehat{\mathcal{A}}$ containing $\mathcal{A}$. The σ-algebra $\widehat{\mathcal{A}}$ obtained in this way is automatically complete in the following sense. If $E \in \widehat{\mathcal{A}}$ with $\widehat{\mu}(E) = 0$ and if $B \subseteq E$, then $B \in \widehat{\mathcal{A}}$ and $\widehat{\mu}(B) = 0$.

To prove the last assertion, let A be an arbitrary subset of X and use part **(c)** of the lemma to note that

$$\widehat{\mu}(A) = \widehat{\mu}(E) + \widehat{\mu}(A) \geq \widehat{\mu}(A \cap B) + \widehat{\mu}(A \setminus B) .$$

Now, as before, the inequality

$$\widehat{\mu}(A) \leq \widehat{\mu}(A \cap B) + \widehat{\mu}(A \setminus B)$$

follows from part **(e)** of the lemma. Therefore B is $\widehat{\mu}$ measurable and

$$0 \leq \widehat{\mu}(B) \leq \widehat{\mu}(E) \leq 0 .$$

Thus $\widehat{\mu}(B) = 0$ as desired.

Next we show that, in case μ is a σ-finite measure, then it has a unique extension to a measure on $\widehat{\mathcal{A}}$.

Theorem (The Hahn Extension Theorem): Assume that μ is a σ-finite measure on an algebra $\mathcal{A}$. Then there exists a unique extension $\widehat{\mu}$ of μ to a measure on $\widehat{\mathcal{A}}$.

Proof: The fact that $\widehat{\mu}$ is a measure on $\widehat{\mathcal{A}}$ was proved in the theorem above—even without the σ-finiteness hypothesis. To prove the uniqueness, let ν be a measure on $\widehat{\mathcal{A}}$ which agrees with μ on $\mathcal{A}$.

Consider first the case that μ and hence $\widehat{\mu}$ are finite measures. Let $E \in \widehat{\mathcal{A}}$ and let $\{E_j\}_{j=1}^{\infty}$ be a collection of elements of $\mathcal{A}$ such that $E \subseteq \cup_j E_j$. Since ν is a measure and agrees with μ on $\mathcal{A}$, we see that

$$\nu(E) \leq \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu(E_j)\,.$$

Thus $\nu(E) \leq \widehat{\mu}(E)$ for any $E \in \widehat{\mathcal{A}}$. Since $\widehat{\mu}$ and ν are additive, we find that $\widehat{\mu}(E) + \widehat{\mu}(X \setminus E) = \nu(E) + \nu(X \setminus E)$. Since the terms on the righthand side are finite and not greater than the corresponding terms on the lefthand side, we conclude that $\widehat{\mu}(E) = \nu(E)$ for all $E \in \widehat{\mathcal{A}}$. This establishes the uniqueness when μ is a finite measure.

We leave the case of σ-finite μ for the reader to study in Bartle's book. $\square$