Figure: This is your instructor.
The Lebesgue Integral
The next step in our program is to show that, if $\mathcal{A}$ is an algebra of subsets of a set $X$ and if $\mu$ is a measure defined on $\mathcal{A}$, then there exists a $\sigma$-algebra $\hat{\mathcal{A}}$ containing $\mathcal{A}$ and a measure $\hat{\mu}$ defined on $\hat{\mathcal{A}}$ so that $\hat{\mu}(E) = \mu(E)$ for $E \in \mathcal{A}$. What we are saying then is that the measure $\mu$ can be extended from the algebra $\mathcal{A}$ to a measure on a $\sigma$-algebra $\hat{\mathcal{A}}$ which contains $\mathcal{A}$.

**Definition:** Let $\mathcal{A}$ be an algebra of sets on $X$. Let $\mu$ be a measure on $\mathcal{A}$. If $F$ is an arbitrary subset of $X$, then we define

$$\hat{\mu}(F) = \inf \sum_{j=1}^{\infty} \mu(E_j),$$

where the infimum is extended over all sequences $\{E_j\}$ of sets in $\mathcal{A}$ such that

$$F \subseteq \bigcup_{j=1}^{\infty} E_j.$$
The function \( \hat{\mu} \) that we have just defined is called the outer measure generated by \( \mu \). This terminology is a bit confusing, just because \( \hat{\mu} \) is usually not a measure. It does, however, have some of the properties of a measure.

**Lemma:** The function \( \hat{\mu} \) of the above definition has the following properties:

(a) \( \hat{\mu}(\emptyset) = 0 \);
(b) \( \hat{\mu}(F) \geq 0 \) for \( F \subseteq X \);
(c) If \( F \subseteq G \), then \( \hat{\mu}(F) \leq \hat{\mu}(G) \);
(d) If \( F \in \mathcal{A} \), then \( \hat{\mu}(F) = \mu(F) \);
(e) If \( \{F_j\} \) is a sequence of subsets of \( X \), then

\[
\hat{\mu}\left(\bigcup_{j=1}^{\infty} F_j\right) \leq \sum_{j=1}^{\infty} \hat{\mu}(F_j).
\]
Proof: Statements (a), (b), and (c) are immediate consequences of the definition.

For (d), note that since \( \{ F, \emptyset, \emptyset, \ldots \} \) is a countable collection of sets in \( \mathcal{A} \) whose union contain \( F \), we see that

\[
\hat{\mu}(F) \leq \mu(F) + 0 + 0 + \cdots = \mu(F).
\]

Conversely, if \( \{ E_j \} \) is any sequence of elements of \( \mathcal{A} \) with \( F \subseteq \cup_j E_j \), then \( F = \cup_j (F \cap E_j) \). Since \( \mu \) is assumed to be a measure on \( \mathcal{A} \), we see that

\[
\mu(F) \leq \sum_{j=1}^{\infty} \mu(F \cap E_j) \leq \sum_{j=1}^{\infty} \mu(E_j),
\]

from which we infer that \( \mu(F) \leq \hat{\mu}(F) \).
To prove (e), let $\epsilon > 0$ and for each $j$ choose a sequence $\{E_{jk}\}$ of sets in $\mathcal{A}$ so that

$$F_j \subseteq \bigcup_{k=1}^{\infty} E_{jk} \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(E_{jk}) \leq \hat{\mu}(F_j) + \frac{\epsilon}{2^j}.$$ 

Since $\{E_{jk} : j, k \in \mathbb{N}\}$ is a countable collection of elements of $\mathcal{A}$ whose union contains $\bigcup F_j$, it follows from the definition of $\hat{\mu}$ that

$$\hat{\mu} \left( \bigcup_{j=1}^{\infty} F_j \right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(E_{jk}) \leq \sum_{j=1}^{\infty} \hat{\mu}(F_j) + \epsilon.$$ 

Since $\epsilon$ is arbitrary, we have obtained the desired inequality. □
Property \((e)\) of the lemma is usually described by the phrase “\(\hat{\mu}\) is countably subadditive.”

**Remark:** An important mathematical point must be made now. The outer measure \(\hat{\mu}\) assigns a length or measure to every subset of \(X\). But of course we know from our considerations in Chapter 1 that that is not possible for a measure. So what is going on here?

The answer is that \(\hat{\mu}\) is not countably additive. In fact it is not necessarily finitely additive. What we shall do in our work below is to restrict \(\hat{\mu}\) to a smaller \(\sigma\)-algebra on which the outer measure is countably additive. The key result here is due to Constantin Carathéodory.
Definition: A subset $E$ of $X$ is said to be $\hat{\mu}$-measurable if

$$\hat{\mu}(A) = \hat{\mu}(A \cap E) + \hat{\mu}(A \setminus E)$$

for all subsets $A \subseteq X$. The collection of all $\hat{\mu}$-measurable sets is denoted by $\hat{\mathcal{A}}$. 
The above condition defines an additivity property on \( \hat{\mu} \). Roughly speaking, a set \( E \) is \( \hat{\mu} \)-measurable in case it and its complement are separated enough so that they divide an arbitrary set \( A \) in an additive fashion.

**Theorem (Carathéodory Extension Theorem):** The collection \( \hat{\mathcal{A}} \) of all \( \hat{\mu} \)-measurable sets is a \( \sigma \)-algebra containing \( \mathcal{A} \). Moreover, if \( \{ E_j \} \) is a pairwise disjoint sequence in \( \hat{\mathcal{A}} \), then

\[
\hat{\mu} \left( \bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \hat{\mu}(E_j).
\]
Proof: Plainly ∅ and X are \( \hat{\mu} \)-measurable. Also, if \( E \in \hat{\mathcal{A}} \), then the complement \( X \setminus E \) is also in \( \hat{\mathcal{A}} \).

For our first step of the proof, we shall show that \( \hat{\mathcal{A}} \) is closed under intersection. Suppose that \( E, F \) are \( \hat{\mu} \)-measurable. Then, for any \( A \subseteq X \),

\[
\hat{\mu}(A \cap F) = \hat{\mu}(A \cap F \cap E) + \hat{\mu}((A \cap F) \setminus E).
\]

Since \( F \in \hat{\mathcal{A}} \), we have

\[
\hat{\mu}(A) = \hat{\mu}(A \cap F) + \hat{\mu}(A \setminus F).
\]
Let $B = A \setminus (E \cap F)$. Then clearly $B \cap F = (A \cap F) \setminus E$ and $B \setminus F = A \setminus F$. Since $F \in \hat{A}$, we see that

$$\hat{\mu}(A \setminus (E \cap F)) = \hat{\mu}((A \cap F) \setminus E) + \hat{\mu}(A \setminus F).$$

Combining the last three equalities gives

$$\hat{\mu}(A) = \hat{\mu}(A \cap (E \cap F)) + \hat{\mu}(A \setminus (E \cap F)).$$

This proves that $E \cap F \in \hat{A}$. Since $\hat{A}$ is closed under intersection and complementation, we may conclude that $\hat{A}$ is an algebra.
Now assume that $E, F \in \hat{A}$ and that $E \cap F = \emptyset$. If we take $A$ to be instead $A \cap (E \cup F)$ in the line above, we obtain

$$\hat{\mu}(A \cap (E \cup F)) = \hat{\mu}(A \cap E) + \hat{\mu}(A \cap F).$$

Letting now $A = X$, we see that $\hat{\mu}$ is additive on $\hat{A}$.

Now we show that $\hat{A}$ is a $\sigma$-algebra and that $\hat{\mu}$ is countably additive on $\hat{A}$. Let $\{E_j\}_{j=1}^{\infty}$ be a pairwise disjoint collection in $\hat{A}$. Set $E = \bigcup_j E_j$. From the previous paragraph we know that $F_k = \bigcup_{j=1}^{k} E_j$ belongs to $\hat{A}$.

Also, if $A$ is any subset of $X$, then

$$\hat{\mu}(A) = \hat{\mu}(A \cap F_k) + \hat{\mu}(A \setminus F_k) = \left[ \sum_{j=1}^{k} \hat{\mu}(A \cap E_j) \right] + \hat{\mu}(A \setminus F_k).$$
Since $F_k \subseteq E$, we may see that $A \setminus E \subseteq A \setminus F_k$. Letting $k \to \infty$ we may conclude from the above that

$$\sum_{j=1}^{\infty} \hat{\mu}(A \cap E_j) + \hat{\mu}(A \setminus E) \leq \hat{\mu}(A).$$

It follows now from the lemma above that

$$\hat{\mu}(A \cap E) \leq \sum_{j=1}^{\infty} \hat{\mu}(A \cap E_j)$$

and

$$\hat{\mu}(A) \leq \hat{\mu}(A \cap E) + \hat{\mu}(A \setminus E).$$
Combining the last three inequalities yields

\[ \hat{\mu}(A) = \hat{\mu}(A \cap E) + \hat{\mu}(A \setminus E) = \sum_{j=1}^{\infty} \hat{\mu}(A \cap E_j) + \hat{\mu}(A \setminus E). \]

This shows in particular that \( E = \bigcup_{j=1}^{\infty} E_j \) is \( \hat{\mu} \)-measurable. Taking \( A = E \), we obtain the desired equation.

The last thing that we must do is to show that \( A \subseteq \hat{A} \). It was proved in part (d) of the lemma that, if \( E \in A \), then \( \hat{\mu}(E) = \mu(E) \). It remains to show that \( E \) is \( \hat{\mu} \)-measurable. So let \( A \) be an arbitrary subset of \( X \). Using part (e) of the lemma, we see that

\[ \hat{\mu}(A) \leq \hat{\mu}(A \cap E) + \hat{\mu}(A \setminus E). \]
To prove the opposite inequality, let $\epsilon > 0$ and let $\{F_j\}$ be a sequence in $A$ such that $A \subseteq \bigcup j F_j$ and

$$\sum_{j=1}^{\infty} \mu(F_j) \leq \hat{\mu}(A) + \epsilon.$$ 

Since $A \cap E \subseteq \bigcup j (F_j \cap E)$ and $A \setminus E \subseteq \bigcup j (F_j \setminus E)$, we may infer from part (e) of the lemma that

$$\hat{\mu}(A \cap E) \leq \sum_{j=1}^{\infty} \mu(F_j \cap E),$$

and

$$\hat{\mu}(A \setminus E) \leq \sum_{j=1}^{\infty} \mu(F_j \setminus E).$$

Thus we may conclude that

$$\hat{\mu}(A \cap E) + \hat{\mu}(A \setminus E) \leq \sum_{j=1}^{\infty} \left\{ \mu(F_j \cap E) + \mu(F_j \setminus E) \right\}$$

$$= \sum_{j=1}^{\infty} \mu(F_j).$$
Since $\epsilon$ was arbitrary, we have proved the desired inequality and the set $E$ belongs to $\hat{A}$. \hfill \Box

Remark: The Carathéodory extension theorem shows that a measure $\mu$ on an algebra $A$ can always be extended to a measure $\hat{\mu}$ on a $\sigma$-algebra $\hat{A}$ containing $A$. The $\sigma$-algebra $\hat{A}$ obtained in this way is automatically complete in the following sense. If $E \in \hat{A}$ with $\hat{\mu}(E) = 0$ and if $B \subseteq E$, then $B \in \hat{A}$ and $\hat{\mu}(B) = 0$.

To prove the last assertion, let $A$ be an arbitrary subset of $X$ and use part (c) of the lemma to note that

$$\hat{\mu}(A) = \hat{\mu}(E) + \hat{\mu}(A) \geq \hat{\mu}(A \cap B) + \hat{\mu}(A \setminus B).$$

Now, as before, the inequality

$$\hat{\mu}(A) \leq \hat{\mu}(A \cap B) + \hat{\mu}(A \setminus B)$$

follows from part (e) of the lemma. Therefore $B$ is $\hat{\mu}$ measurable and

$$0 \leq \hat{\mu}(B) \leq \hat{\mu}(E) \leq 0.$$  

Thus $\hat{\mu}(B) = 0$ as desired.
Next we show that, in case $\mu$ is a $\sigma$-finite measure, then it has a unique extension to a measure on $\hat{A}$.

**Theorem (The Hahn Extension Theorem):** Assume that $\mu$ is a $\sigma$-finite measure on an algebra $\mathcal{A}$. Then there exists a unique extension $\hat{\mu}$ of $\mu$ to a measure on $\hat{A}$.
Proof: The fact that $\hat{\mu}$ is a measure on $\hat{A}$ was proved in the theorem above—even without the $\sigma$-finiteness hypothesis. To prove the uniqueness, let $\nu$ be a measure on $\hat{A}$ which agrees with $\mu$ on $A$.

Consider first the case that $\mu$ and hence $\hat{\mu}$ are finite measures. Let $E \in \hat{A}$ and let $\{E_j\}_{j=1}^{\infty}$ be a collection of elements of $A$ such that $E \subseteq \bigcup_j E_j$. Since $\nu$ is a measure and agrees with $\mu$ on $A$, we see that

$$
\nu(E) \leq \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu(E_j).
$$

Thus $\nu(E) \leq \hat{\mu}(E)$ for any $E \in \hat{A}$. Since $\hat{\mu}$ and $\nu$ are additive, we find that $\hat{\mu}(E) + \hat{\mu}(X \setminus E) = \nu(E) + \nu(X \setminus E)$. Since the terms on the righthand side are finite and not greater than the corresponding terms on the lefthand side, we conclude that $\hat{\mu}(E) = \nu(E)$ for all $E \in \hat{A}$. This establishes the uniqueness when $\mu$ is a finite measure.
We leave the case of $\sigma$-finite $\mu$ for the reader to study in Bartle’s book.