Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Construction of Lebesgue Measure

Up until this point in the course we have taken the existence of Lebesgue measure—as an extension of the notion of length of an interval—for granted. Now we have the machinery developed to actually construct Lebesgue measure. And we do so.

Refer now to our lemma from last time. We learned there that the collection $\mathcal{F}$ of all intervals is an algebra of subsets of $\mathbb{R}$. Also the length function ℓ gives a measure on this algebra. If we apply the Carathéory extension theorem to this ℓ and $\mathcal{F}$, then we generate a measure space $(X, \mathcal{F}^*, \ell^*)$. The σ-algebra $\mathcal{F}^*$ is called the collection of *Lebesgue measurable sets* and the measure ℓ^* on $\mathcal{F}^*$ is called *Lebesgue measure.*

Although it is sometimes useful to work with $(X, \mathcal{F}^*, \ell^*)$, it is often more convenient to deal with the smallest σ-algebra containing $\mathcal{F}$ rather than with $\mathcal{F}^*$. And in fact that smallest σ-algebra is the collection $\mathcal{B}$ of Borel sets. The restriction of Lebesgue measure to the σ-algebra $\mathcal{B}$ is still called Lebesgue measure. It is a fact that every Lebesgue measurable set is contained in a Borel set with the same measure. And every Lebesgue measurable function is equal almost everywhere to a Borel measurable function. [We will verify these assertions below.] So there is little loss of generality to specialize down to the σ-algebra of Borel sets.

In this brief section we describe a generalization of the construction in the last lecture. The new measure that we construct here is useful in probability theory and other applications.

Let $g : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. We assume that g is right continuous at each point. We also suppose that $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$ exist. The values of the latter two limits could in fact be $-\infty$, $+\infty$ respectively.

Now we define

$$\begin{aligned}
\mu_g((a,b]) &= g(b) - g(a)\,, \\
\mu_g((-\infty,b]) &= g(b) - \lim_{x\to-\infty} g(x)\,, \\
\mu_g((a,+\infty)) &= \lim_{x\to+\infty} g(x) - g(a)\ , \\
\mu_g((-\infty,+\infty)) &= \lim_{x\to+\infty} g(x) - \lim_{x\to-\infty} g(x)\,.
\end{aligned}$$

We go on to define μ_g on the algebra $\mathcal{F}$ (where $\mathcal{F}$ is as in the last section) of finite pairwise disjoint unions of such sets to be the corresponding sums. It is easy to see that this μ_g gives a σ-finite measure on the algebra $\mathcal{F}$.

As a result, this measure has a unique extension, which we also denote by μ_g, to the algebra of all Borel subsets of $\mathbb{R}$. This extension is often called the *Borel-Stieltjes measure* generated by g. According to an earlier theorem, μ_g actually has an extension to a complete σ-algebra which contains the Borel sets. This last extension is called the *Lebesgue-Stieltjes measure* generated by g. We encountered these ideas earlier in Chapter 2 of the book.

Let X be a compact Hausdorff space. It is useful to know the dual of $C(X)$, the space of continuous functions on X under the supremum norm. We treat this matter in the present discussion. In fact it will turn out that the Banach space dual of $C([0, 1])$ is a space of measures.

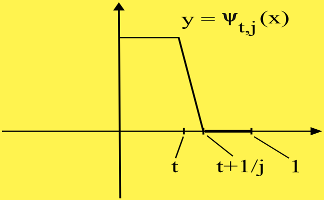
The main result here is, like two of our earlier results, called the Riesz representation theorem.

Theorem (Riesz representation theorem, third version): Let $I = [0,1]$. Let φ be a positive, bounded linear functional on $C(I)$. Then there exists a measure γ defined on the Borel subsets of I such that

$$\varphi(f) = \int_I f \, d\gamma$$

for all $f \in C(I)$. Furthermore, the norm $\|\varphi\|$ of φ equals $\gamma(I)$.

Proof: If $t \in \mathbb{R}$ is such that $0 \leq t < 1$ and j is a sufficiently large natural number, then let $\psi_{t,j}$ be the function in $C(I)$ which equals 1 on $[0, t]$ and which equals 0 on $(t + 1/j, 1]$. We also specify that $\psi_{t,j}$ be linear on $(t, t + 1/j)$. See the figure.



If $j \leq k$ and $x \in I$, then $0 \leq \psi_{t,k}(x) \leq \psi_{t,j}(x) \leq 1$ so that the sequence $\{\varphi(\psi_{t,j})\}$ is bounded and decreasing. If $t \in [0, 1)$, then we define

$$g(t) = \lim_{j \to \infty} \varphi(\psi_{t,j}) .$$

Set $g(t) = 0$ for $t < 0$. Define $\psi_1(x) \equiv 1$ for all $x \in I$. If $t \geq 1$, we set $g(t) = \varphi(\psi_1)$. Then g is a monotone increasing function on $\mathbb{R}$.

We claim that g is right continuous. This statement is clear if $t < 0$ or $t \geq 1$. Let $t \in [0, 1)$ and $\epsilon > 0$.

Throughout this proof we use the supremum norm on continuous functions. Choose

$$j > \max\{2, \|\varphi\| \cdot \epsilon^{-1}\}$$

to be so large that

$$g(t) \le \varphi(\psi_{t,j}) \le g(t) + \epsilon \, .$$

If ρ_j is the function in $C(I)$ which equals 1 on $[0, t + j^{-2}]$, which equals 0 on $(t + j^{-1} - j^{-2}, 1]$, and which is linear on

$$(t + j^{-2}, t + j^{-1} - j^{-2}] \,,$$

then it is straightforward to check that $\|\rho_j - \psi_{t,j}\| \leq 1/j$. Therefore

$$\varphi(\rho_j) \leq \varphi(\psi_{t,j}) + \left(\frac{1}{j}\right) \|\varphi\| \leq g(t) + 2\epsilon \,.$$

It follows that $g(t) \leq g(t + j^{-2}) \leq g(t) + 2\epsilon$.

By the Hahn extension theorem, there exists a measure γ on the Borel subsets of $\mathbb{R}$ so that $\gamma([\alpha, \beta]) = g(\beta) - g(\alpha)$. This shows that $\gamma(E) = 0$ if $E \cap I = \emptyset$. It also implies that

$$\gamma([0, c]) = \gamma((-1, c]) = g(c)$$

and that $\|\varphi\| = |\varphi(\psi_1)| = g(1) = \gamma(I)$.

The last thing we must do is to show that the linear functional φ is represented by integration against the measure γ. Let $\epsilon > 0$. Since f is uniformly continuous on I, there is a $\delta > 0$ so that, if $|x - y| < \delta$ and $x, y \in I$, then $|f(x) - f(y)| < \epsilon$. Let

$$a = t_0 < t_1 < \cdots < t_m = b$$

be such that $\max\{t_k - t_{k-1}\} < \delta/2$. Choose j so large that $2/j < \min\{t_k - t_{k-1}\}$ and such that, for $k = 1, 2, \ldots, m$, we have

$$g(t_k) \leq \varphi(\psi_{t_k,j}) \leq g(t_k) + \epsilon(m\|f\|)^{-1} .$$

Now we consider functions defined on I by

$$f_1(x) = f(t_1) \cdot \psi_{t_1,j}(x) + \sum_{k=2}^{m} f(t_k)\{\psi_{t_k,j}(x) - \psi_{t_{k-1},j}(x)\} ,$$

$$f_2(x) = f(t_1)\chi_{[t_0,t_1]}(x) + \sum_{k=2}^{m} f(t_k)\chi_{(t_{k-1},t_k]}(x) .$$

Notice that $f_1 \in C(I)$ and that f_2 is a step function on I. Clearly $\sup_{x\in I} |f_2(x) - f(x)| \leq \epsilon$. As an exercise, the reader should also show that $\|f_1 - f\| \leq \epsilon$.

Thus we have

$$|\varphi(f) - \varphi(f_1)| \le \epsilon \|\varphi\| \, .$$

We see that, if $2 \le k \le m$, then

$$|\varphi(\psi_{t_k,j} - \psi_{t_{k-1},j}) - \{g(t_k) - g(t_{k-1})\}| \le \epsilon (m\|f\|)^{-1} \, .$$

Now we apply φ to f_1 and integrate f_2 with respect to γ. The last inequality then gives us

$$|\varphi(f_1) - \int_I f_2 \, d\gamma| \le \epsilon \, .$$

Since f_2 lies within ϵ of f, we also have

$$\left| \int_I f_2 \, d\gamma - \int_I f \, d\gamma \right| \leq \epsilon \gamma(I) \, .$$

Combining the inequalities, we finally obtain

$$\left| \varphi(f) - \int_I f \, d\gamma \right| \leq \epsilon (2\|\varphi\| + 1) \, .$$

Because $\epsilon > 0$ is arbitrary, we finally deduce that $\varphi(f)$ is given by integration of f against γ. $\square$

Remark: If the reader will check the proof of the theorem, it will be seen that an arbitrary bounded linear functional φ on $C(I)$ can be written as the difference $\varphi^+ - \varphi^-$ of two positive, bounded linear functionals. Using this observation, we can extend this latest Riesz representation theorem so that we may represent any bounded linear functional on $C(I)$ by means of integration with respect to a signed measure defined on the Borel subsets of I.