Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Particular Sets

What we have done in the last several chapters is to construct a σ-algebra $\mathcal{L}$ of subsets of $\mathbb{R}$, called the *Lebesgue measurable sets*. And we have also constructed *Lebesgue measure*, which is a function whose domain is $\mathcal{L}$.

Certainly $\mathcal{L}$ includes the intervals, but in fact $\mathcal{L}$ is quite large and contains a great variety of sets. It is this last point that we address in the current discussion.

Figure: The structure of an open set.

Lemma: *Let U be an open set in $\mathbb{R}$. Then U is the countable pairwise disjoint union of open intervals. See the figure.*

Proof: Define a relation on U by $x \sim y$ if all the real numbers between x and y also lie in U. You can check that this is an equivalence relation.

Let E be an equivalence class from this relation. Then E is an interval. For if $a, b \in E$ then all the points between a and b must lie in E by the definition of the relation. And in fact E is an open interval by similar reasoning.

Of course the equivalence classes are pairwise disjoint. That ends the proof. □

Proposition: *Every open set in $\mathbb{R}$ is Lebesgue measurable. Every closed set in $\mathbb{R}$ is Lebesgue measurable.*

Proof: We know that each open interval is Lebesgue measurable. And countable unions of measurable sets are measurable. Thus it follows from the preceding lemma that every open set is Lebesgue measurable.

That every closed set is Lebesgue measurable now follows by complementation. □

Definition: Let E be the intersection of a countable collection of open sets. Then E is called a G_δ.

Definition: Let F be the union of a countable collection of closed sets. Then F is called an F_σ.

Example: Let $I_j = (-1/j, 1/j)$. Then each I_j is open, so $E = \cap_j I_j$ is a G_δ. But notice that $E = \{0\}$ is *not* open.

Let $J_j = [1/j, 1 - 1/j]$. Then each J_j is closed, so $F = \cup_j J_j$ is an F_σ. But notice that $F = (0, 1)$ is *not* closed.

Example: A set that is the union of countably many G_δ sets is called a $G_{\delta\sigma}$ set. A set that is the intersection of countably many F_σ sets is called an $F_{\sigma\delta}$ set. Of course all these sets are Lebesgue measurable.

Definition: The smallest σ-algebra that contains all the open intervals is called the *Borel σ-algebra $\mathcal{B}$*.

We can think of the "smallest σ-algebra that contains all the open intervals" as the intersection of all the σ-algebras that contain the open intervals.

It is natural to observe that $\mathcal{B} \subseteq \mathcal{L}$. In fact the two σ-algebras are *not* equal, and we shall prove this assertion later in our discussions. The proof that we give will be abstract and non-constructive. But, at the very end of the course, we actually construct a Lebesgue measurable set that is not Borel.