Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Lebesgue Null Sets

We now treat sets that are small in the sense of Lebesgue measure but quite significant for our theory. These are the Lebesgue null sets.

Definition: A set $E \subseteq \mathbb{R}$ is called a *Lebesgue null set* if $m^*(E) = 0$.

Example: Of course the empty set $\emptyset$ is a Lebesgue null set.

It is obvious that any finite set has measure zero.

In point of fact any countable set is a Lebesgue null set. To see this, let $\{a_j\}$ be a countable set. Let $\epsilon > 0$ and let

$$I_j = (a_j - 2^{-j-1}\epsilon, a_j + 2^{-j-1}\epsilon) \, .$$

Then

$$E \subseteq \bigcup_{j=1}^{\infty} I_j$$

so that

$$m^*(E) \le \sum_{j=1}^{\infty} m(I_j) \le \sum_{j=1}^{\infty} 2^{-j}\epsilon = \epsilon \, .$$

Since this is true for any $\epsilon > 0$, we see that $m^*(E) = 0$.

It is common to summarize the last example by saying that any *denumerable set* has measure 0.

Proposition: *Any null set Z is Lebesgue measurable. It follows that $m(Z) = 0$. Furthermore, any subset of Z is Lebesgue measurable and a null set.*

Proof: Let $A \subseteq \mathbb{R}$ be any set. Since $Z \supseteq A \cap Z$ and $A \supseteq A \cap {}^cZ$, we may conclude from the monotonicity of m^* that

$$m^*(A) = m^*(Z) + m^*(A) \geq m^*(A \cap Z) + m^*(A \cap {}^cZ) .$$

As a result, our measurability lemma tells us that Z is Lebesgue measurable. Therefore $m(Z) = m^*(Z) = 0$.

If W is a subset of Z, then $0 \leq m^*(W) \leq m^*(Z)$ so that W is also a null set. Hence W is Lebesgue measurable. $\square$

The property that any subset of a Lebesgue null set is measurable and has measure zero is sometimes summarized by saying that Lebesgue measure is *complete*.

It is natural to think that a Lebesgue null set cannot have too many points. But this would be a misconception.

Proposition: *There exist Lebesgue null sets that have uncountably many points.*

Proof: Certainly the Cantor ternary a closed, indeed a compact, subset of $\mathbb{R}$. The construction of the Cantor set shows that the complement of the Cantor set in the unit interval has length 1. Hence the Cantor set is a Lebesgue null set. But we know that the Cantor set has uncountably many points. $\square$

Invariance under Translation

An important feature of Lebesgue measure is that it is translation invariant. Intuitively speaking, this means that if we take a measurable set $E \subseteq \mathbb{R}$ and translate it by adding a constant a to each element of the set,

$$E_a = \{e + a : e \in E\},$$

then the measure of the set is unchanged. In other words, $m(E) = m(E_a)$. The present section develops this collection of ideas.

Proposition: *With E and E_a defined as above, $m(E) = m(E_a)$.*
Proof: Let A and B be arbitrary subsets of $\mathbb{R}$. Let $a \in \mathbb{R}$. Then it is easy to check that

$$A_a \cap B = [A \cap B_{-a}]_a \ .$$

Similarly, one can show that

$$[{}^c B]_a = {}^c\,[B_a] \ .$$

Now let $b = -a$. It follows from the obvious invariance of m^* under translation that

$$m^*(A_a \cap B) = m^*(A \cap B_b) .$$

We let $E \in \mathcal{L}$ and use our measurability equation with $B = E$ and also with $B = {}^cE$ to obtain

$$\begin{aligned} m^*(A) &= m^*(A_a) \\ &= m^*((A_a) \cap E) + m^*((A_a) \cap {}^cE) \\ &= m^*(A \cap E_b) + m^*(A \cap ({}^cE)_b) \\ &= m^*(A \cap E_b) + m^*(A \cap {}^c(E_b)) \end{aligned}$$

for all $A \subseteq \mathbb{R}$. Thus E_b is also Lebesgue measurable.

Now it is straightforward to check that m^* is translation invariant—just because the notion of length of an interval is translation invariant. Thus we see that

$$m(E_b) = m^*(E_b) = m^*(E) = m(E) . \qquad \square$$

A Lebesgue Measurable set that is Not Borel

We now show that there exists a Lebesgue measurable set that is not Borel. Our treatment of this topic will rely on Cantor's notion of cardinality.

Theorem: *There exists a Lebesgue measurable set of real numbers which is not Borel.*

Proof: Note that there are a countable number of open intervals in $\mathbb{R}$ with both endpoints rational. Call this cardinality $\mathcal{N}_0$. It is straightforward to see that $\mathcal{B}$ is the smallest σ algebra that contains these intervals. Thus the cardinality of $\mathcal{B}$ is

$$(\mathcal{N}_0)^{\mathcal{N}_0} = c\,.$$

Here c is the cardinality of the real numbers $\mathbb{R}$.

Now we know that $\mathcal{L}$ contains a null set with uncountably many elements. Since an arbitrary subset of a null set is still a null set, we may conclude that $\mathcal{L}$ contains at least 2^{c} elements. Thus

$$\operatorname{card}(\mathcal{B}) = \mathsf{c} < 2^{\mathsf{c}} \leq \operatorname{card}(\mathcal{L}) .$$

It follows then that $\mathcal{B}$ is a proper subset of $\mathcal{L}$. $\square$

Remark: It is actually possible to explicitly construct a Lebesgue measurable set that is not Borel. For the details, and some of the history, refer to Kanovei.

At the very end of the course we treat yet another method for constructing a Lebesgue measurable set that is not Borel.