Figure: This is your instructor.
The Lebesgue Integral
Now we consider the idea of approximating a Lebesgue measurable set from the inside by closed sets.

**Theorem:** A set $S \subseteq \mathbb{R}$ is Lebesgue measurable if and only if, for each $\epsilon > 0$, there is a closed set $F \subseteq S$ with $m^*(S \setminus F) < \epsilon$.

**Proof:** If $S$ is measurable, then its complement $^cS$ is also measurable. By the proposition above, there is an open set $\Omega$ with $^cS \subseteq \Omega$ and $m(\Omega \setminus ^cS) < \epsilon$. Now we let $F = ^c\Omega$. Thus $F$ is closed. Also $F \subseteq S$. Further note that $S \setminus F = S \cap \Omega = \Omega \setminus ^cS$. Hence

$$m(S \setminus F) = m(\Omega \setminus ^cS) < \epsilon.$$
For the converse direction, note that for each \( j \in \mathbb{N} \) there is a closed set \( F_j \subseteq S \) with \( m^*(S \setminus F_j) < 1/j \). Let \( K = \bigcup_{j=1}^{\infty} F_j \). Then \( K \) is an \( F_\sigma \) set. So it is Lebesgue measurable. Since \( F_j \subseteq K \), we see that \( S \setminus K \subseteq S \setminus F_j \). Hence

\[
m^*(S \setminus K) \leq m^*(S \setminus F_j) < \frac{1}{j}
\]

for all \( j \). Thus \( m^*(S \setminus K) = 0 \). This implies that \( W = S \setminus K \) is a measurable set. In conclusion, \( S = K \cup W \) is measurable. \( \square \)
Corollary: Let $S \subseteq \mathbb{R}$ be Lebesgue measurable. Then, for any $\epsilon > 0$, there is a closed set $F \subseteq S$ with $m(S) \leq m(F) + \epsilon$. Thus we have

$$m(S) = \sup \{ m(F) : F \text{ closed and } F \subseteq S \}.$$
Proof: The set $F$ in the theorem is measurable and $F \subseteq S$. Thus

$$m(S) = m^*(S \cap F) + m^*(S \setminus F) \leq m(F) + \epsilon.$$
Next we characterize measurability in terms of approximation by $F_\sigma$ sets from the inside.

**Corollary:** The following statements are equivalent for a set $E \subseteq \mathbb{R}$.

(a) The set $E$ is Lebesgue measurable.
(b) There exists an $F_\sigma$ set $K$ with $K \subseteq E$ and $m^*(E \setminus K) = 0$.
(c) There is an $F_\sigma$ set $K$ and a Lebesgue null set $W$ such that $K \subseteq E$, $W \subseteq E$, and $E = K \cup W$.

**Proof:** The proof is a straightforward application of the preceding ideas. We leave the details to the reader.

\[\square\]
Of course a compact set, being bounded, always has finite Lebesgue measure. So it is of interest that we can approximate \textit{virtually any} Lebesgue measurable set from within by compact sets.

\textbf{Figure:} Approximation from the inside by a compact set.
Theorem: A set $E \subseteq \mathbb{R}$ with $m^*(E) < \infty$ is Lebesgue measurable if and only if, for each $\epsilon > 0$, there is a compact set $K$ such that $K \subseteq E$ and $m^*(E \setminus K) < \epsilon$. See the figure.

Proof: If $E$ is measurable and $j \in \mathbb{N}$, then we let $E_j = E \cap \{x : |x| \leq j\}$. Since the sequence of sets $E_j$ increases to $E$, we may conclude that the numerical sequence $\{m(E_j)\}$ increases to $m(E)$. Hence there is an index $j_0$ so that $m(E) < m(E_{j_0}) + \epsilon/2$. By the theorem above there is a closed set $F$ with $F \subseteq E_{j_0}$ and $m(E_{j_0} \setminus F) < \epsilon/2$. 
Of course $E$ is the disjoint union of the sets $E \setminus E_{j_0}$ and $E_{j_0}$ so we have

$$m(E) = m(E \setminus E_{j_0}) + m(E_{j_0}).$$

Since $m(E) < \infty$, we also know that

$$m(E \setminus E_{j_0}) = m(E) - m(E_{j_0}) < \frac{\epsilon}{2}.$$

Furthermore, $E \setminus F$ is the union of the disjoint sets $E \setminus E_{j_0}$ and $E_{j_0} \setminus F$. So we have

$$m(E \setminus F) = m(E \setminus E_{j_0}) + m(E_{j_0} \setminus F) < \epsilon.$$

Since $F \subseteq E_{j_0}$ is closed and bounded, it is compact. That proves the forward direction of the theorem.
For the converse, suppose that for each $j \in \mathbb{N}$ there is a compact set $F_j$ with $F_j \subseteq E$ and $m^*(E \setminus F_j) < 1/j$. We set $F = \bigcup_{j=1}^{\infty} F_j$. Then $F$ is measurable and $E \setminus F \subseteq E \setminus F_j$ for each $j \in \mathbb{N}$. Thus we have $m^*(E \setminus F) = 0$. So $W = E \setminus F$ is a Lebesgue null set and so is measurable. In conclusion, $E = F \cup W$ is Lebesgue measurable. \qed
We conclude this discussion by showing that any Lebesgue measurable set can be approximated by the finite union of bounded, open intervals. In this discussion we use the notation

\[ S \triangle T \equiv (S \setminus T) \cup (T \setminus S). \]
**Theorem:** Let $E \subseteq \mathbb{R}$ have finite Lebesgue measure. Let $\epsilon > 0$. Then there are bounded open intervals $I_1, I_2, \ldots, I_k$ so that, if $U = \bigcup_{j=1}^{k} I_j$, then $m(E \triangle U) < \epsilon$.

**Proof:** Let $\epsilon > 0$. As we have seen before, there is a collection $\{I_j\}_{j=1}^{\infty}$ of bounded, open intervals which cover $E$ and so that, if $\mathcal{I} = \bigcup_{j=1}^{\infty} I_j$, then $m(\mathcal{I}) \leq m(E) + \epsilon/2$. We also know that there is a compact set $K \subseteq E$ such that $m(E \setminus K) < \epsilon/2$. 


The Heine-Borel theorem tells us that finitely many of the $I_j$, say $I_1, I_2, \ldots, I_m$, cover $K$. If we set $L = \bigcup_{j=1}^m I_j$, then $K \subseteq L \subseteq \mathcal{I}$ and $K \subseteq E \subseteq \mathcal{I}$. Therefore

$$m(E \setminus L) = m(E \setminus L) + m(L \setminus E) \leq m(E \setminus K) + m(\mathcal{I} \setminus E) < \epsilon.$$ 

That completes the proof. \hfill \square
You can check for yourself that this last theorem is also true for closed intervals, or half-open intervals, or even for pairwise disjoint intervals.