Figure: This is your instructor.
The Lebesgue Integral
Different Methods of Convergence
Let \((X, \mathcal{X}, \mu)\) be a measure space. In this course we have treated four different types of convergence of a sequence of \(\{f_j\}\) functions to a limit function \(f\):
(i) **pointwise convergence:** For each \( \epsilon > 0 \) and each \( x \in X \) there is a number \( J > 0 \) such that, if \( j > J \), then \( |f_j(x) - f(x)| < \epsilon \).

(ii) **uniform convergence:** For each \( \epsilon > 0 \) there is a number \( J > 0 \) such that, if \( j > J \) and \( x \in X \), then \( |f_j(x) - f(x)| < \epsilon \).

(iii) **convergence almost everywhere:** There exists a set \( E \subseteq X \) with \( \mu(E) = 0 \) so that, for every \( \epsilon > 0 \), and each \( x \in X \setminus E \), there is a number \( J > 0 \) such that, for \( j > J \), we have \( |f_j(x) - f(x)| < \epsilon \).

(iv) **convergence in \( L^p \), \( 1 \leq p < \infty \):** For each \( \epsilon > 0 \) there is a number \( J > 0 \) so that, if \( j > J \), then

\[
\|f_j - f\|_{L^p} = \int |f_j(x) - f(x)|^p \, d\mu(x)^{1/p} < \epsilon.
\]
It is clear that uniform convergence implies pointwise convergence. Also pointwise convergence implies convergence almost everywhere. In the case of a finite measure space, uniform convergence also implies convergence in $L^p$. The reverse implications are false.

**Example:** Let $f_j(x) = \chi_{[j,\infty)}$. Then the $f_j$ converge to the identically 0 function pointwise and almost everywhere, but not in $L^p$ for any $p \geq 1$. They do not converge uniformly.

Let $g_j(x) = \chi_{[1,1+1/j]}$. These functions converge almost everywhere and in $L^p$ to the identically 0 function. They do not converge pointwise, and they do not converge uniformly.
**Proposition:** Let $(X, \mathcal{X}, \mu)$ be a measure space. Assume that $\mu(X) < +\infty$. Let $\{f_j\}$ be a sequence of $L^p$ functions that converges uniformly on $X$ to a limit function $f$. Then $f \in L^p$ and the sequence $\{f_j\}$ converges in $L^p$ to $f$. 
Proof: Let $\epsilon > 0$ and choose a $J > 0$ such that when $j > J$ and $x \in X$, $|f_j(x) - f(x)| < \epsilon$. Observe that, if $j > J$, then

$$
\|f_j - f\|_{L^p} = \left\{ \int |f_j(x) - f(x)|^p \, d\mu \right\}^{1/p} \\
\leq \left\{ \int \epsilon^p \, d\mu \right\}^{1/p} \\
= \epsilon \mu(X)^{1/p}.
$$

(*)

We conclude that $\{f_j\}$ converges in $L^p$ to $f$. \qed
Proposition: Let $(X, \mathcal{X}, \mu)$ be a measure space. Let $1 \leq p < \infty$. Let $\{f_j\}$ be a sequence in $L^p$ which converges pointwise almost everywhere to a measurable function $f$. If there is a $g \in L^p$ such that

$$|f_j(x)| \leq g(x) \quad \forall x \in X, \forall j \in \mathbb{N},$$

then $f \in L^p$ and $f_j \to f$ in $L^p$.

Proof: Because of inequality (*), we see that $|f(x)| \leq g(x)$ almost everywhere. Since $g \in L^p$, we conclude that $f \in L^p$. 
Observe that

\[ |f_j(x) - f(x)|^p \leq [2g(x)]^p, \quad \text{a.e.} \]

Since \( \lim_{j \to \infty} |f_j(x) - f(x)|^p = 0 \) a.e. and \( 2^p g^p \) belongs to \( L^1 \), the Lebesgue dominated convergence theorem tells us that

\[ \lim_{j \to \infty} \int |f_j - f|^p \, d\mu = 0. \]

As a result, \( f_j \to f \) in \( L^p \). \qed
**Corollary:** Let \((X, \mathcal{X}, \mu)\) be a measure space. Assume that 
\(\mu(X) < +\infty\). Let \(1 \leq p < \infty\). Let \(\{f_j\}\) be a sequence in \(L^p\) which converges almost everywhere to a measurable function \(f\). If there is a constant \(K > 0\) such that

\[|f_j(x)| \leq K \quad \forall x \in X, \forall j \in \mathbb{N}, \quad \text{(**)\] 

then \(f\) belongs to \(L^p\) and the sequence \(\{f_j\}\) converges to \(f\) in \(L^p\).
Proof: Since $\mu(X) < +\infty$, then the constant functions belong to $L^p$. So the function $g(x) \equiv K$ is in $L^p$. Now apply the proposition.

One might suppose that $L^p$ convergence implies almost everywhere convergence. But the next example shows that that is not the case.
Example: Let $X = [0, 1]$, $B$ be the Borel sets, and $\mu$ be Lebesgue measure. Consider the intervals in $[0, 1]$ with dyadic\(^1\) endpoints. Order these intervals in decreasing order of size. Let $f_j$ be the characteristic function of the $j$th interval.

Then it is clear that the $f_j$ tend to $f \equiv 0$ in $L^p$ norm. But, if $x$ is any point of $[0, 1]$, then there is a subsequence $f_{j_k}$ that equals 1 at $x$ and there is another subsequence $f_{j_\ell}$ that equals 0 at $x$. So we do \textit{not} have pointwise convergence at \textit{any point} of the interval $[0, 1]$.

\(^1\)A point is dyadic if it has the form $j/2^k$. 

In this section we treat a new concept of convergence which is analytically useful. And it is intuitively appealing.

**Definition:** Let \((X, \mathcal{X}, \mu)\) be a measure space. A sequence \(\{f_j\}\) of measurable functions is said to **converge in measure** to a measurable function \(f\) precisely when

\[
\lim_{j \to \infty} \mu(\{x \in \mathbb{R} : |f_j(x) - f(x)| \geq \alpha\}) = 0
\]

for each \(\alpha > 0\).
The sequence \( \{f_j\} \) is said to be \textit{Cauchy in measure} when

\[
\lim_{j,k \to \infty} \mu(\{x \in \mathbb{R} : |f_j(x) - f_k(x)| \geq \alpha\}) = 0
\]

for each \( \alpha > 0 \).
Example:  Let $f_j(x) = \chi_{[j,\infty)}$. Then the $f_j$ do not converge in measure. Indeed they are not Cauchy in measure.

Let $g_j(x) = \chi_{[1,1+1/j]}$. Then the $g_j$ converge in measure to the identically $0$ function.
Proposition: Let \((X, \mathcal{X}, \mu)\) be a measure space. If the functions \(\{f_j\}\) converge in \(L^p\), \(1 \leq p < \infty\), then the sequence converges in measure.
Proof: Let $\alpha > 0$. Set

$$E_j(\alpha) = \{x \in \mathbb{R} : |f_j(x) - f(x)| \geq \alpha\}.$$ 

Then

$$\int |f_j - f|^p d\mu \geq \int_{E_j(\alpha)} |f_j - f|^p d\mu \geq \alpha^p \cdot \mu(E_j(\alpha)).$$

We know that $\|f_j - f\|_{L^p} \to 0$. Since $\alpha > 0$, we may conclude that $\mu(E_j(\alpha)) \to 0$ as $j \to \infty$. \qed