Math 4121
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Steven G. Krantz

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Figure: This is your instructor.

The Lebesgue Integral

Different Methods of Convergence

Let $(X, \mathcal{X}, \mu)$ be a measure space. In this course we have treated four different types of convergence of a sequence of $\{f_j\}$ functions to a limit function f:

(i) pointwise convergence: For each $\epsilon > 0$ and each $x \in X$ there is a number $J > 0$ such that, if $j > J$, then $|f_j(x) - f(x)| < \epsilon$.

(ii) uniform convergence: For each $\epsilon > 0$ there is a number $J > 0$ such that, if $j > J$ and $x \in X$, then $|f_j(x) - f(x)| < \epsilon$.

(iii) convergence almost everywhere: There exists a set $E \subseteq X$ with $\mu(E) = 0$ so that, for every $\epsilon > 0$, and each $x \in X \setminus E$, there is a number $J > 0$ such that, for $j > J$, we have $|f_j(x) - f(x)| < \epsilon$.

(iv) convergence in L^{p}, $1 \leq \mathrm{p} < \infty$: For each $\epsilon > 0$ there is a number $J > 0$ so that, if $j > J$, then

$$\|f_j - f\|_{L^p} = \int |f_j(x) - f(x)|^p \, d\mu(x)^{1/p} < \epsilon \, .$$

It is clear that uniform convergence implies pointwise convergence. Also pointwise convergence implies convergence almost everywhere. In the case of a finite measure space, uniform convergence also implies convergence in L^p. The reverse implications are false.

Example: Let $f_j(x) = \chi_{[j,\infty)}$. Then the f_j converge to the identically 0 function pointwise and almost everywhere, but not in L^p for any $p \geq 1$. They do not converge uniformly.
Let $g_j(x) = \chi_{[1,1+1/j]}$. These functions converge almost everywhere and in L^p to the identically 0 function. They do not converge pointwise, and they do not converge uniformly.

Proposition: *Let* $(X, \mathcal{X}, \mu)$ *be a measure space. Assume that* $\mu(X) < +\infty$. *Let* $\{f_j\}$ *be a sequence of* L^p *functions that converges uniformly on* X *to a limit function* f. *Then* $f \in L^p$ *and the sequence* $\{f_j\}$ *converges in* L^p *to* f.

Proof: Let $\epsilon > 0$ and choose a $J > 0$ such that when $j > J$ and $x \in X$, $|f_j(x) - f(x)| < \epsilon$. Observe that, if $j > J$, then

$$\begin{aligned} \|f_j - f\|_{L^p} &= \left\{ \int |f_j(x) - f(x)|^p \, d\mu \right\}^{1/p} \\ &\leq \left\{ \int \epsilon^p \, d\mu \right\}^{1/p} \\ &= \epsilon \mu(X)^{1/p} . \end{aligned} \qquad (*)$$

We conclude that $\{f_j\}$ converges in L^p to f. □

Proposition: *Let $(X, \mathcal{X}, \mu)$ be a measure space. Let $1 \leq p < \infty$. Let $\{f_j\}$ be a sequence in L^p which converges pointwise almost everywhere to a measurable function f. If there is a $g \in L^p$ such that*

$$|f_j(x)| \leq g(x) \quad \forall x \in X \ , \quad \forall j \in \mathbb{N} \, ,$$

then $f \in L^p$ and $f_j \to f$ in L^p.

Proof: Because of inequality $(*)$, we see that $|f(x)| \leq g(x)$ almost everywhere. Since $g \in L^p$, we conclude that $f \in L^p$.

Observe that

$$|f_j(x) - f(x)|^p \leq [2g(x)]^p \ , \quad \text{a.e.}$$

Since $\lim_{j\to\infty} |f_j(x) - f(x)|^p = 0$ a.e. and $2^p g^p$ belongs to L^1, the Lebesgue dominated convergence theorem tells us that

$$\lim_{j\to\infty} \int |f_j - f|^p \, d\mu = 0 \, .$$

As a result, $f_j \to f$ in L^p. □

Corollary: *Let $(X, \mathcal{X}, \mu)$ be a measure space. Assume that $\mu(X) < +\infty$. Let $1 \leq p < \infty$. Let $\{f_j\}$ be a sequence in L^p which converges almost everywhere to a measurable function f. If there is a constant $K > 0$ such that*

$$|f_j(x)| \leq K \quad \forall x \in X \ , \ \forall j \in \mathbb{N} \, , \qquad (**)$$

then f belongs to L^p and the sequence $\{f_j\}$ converges to f in L^p.

Proof: Since $\mu(X) < +\infty$, then the constant functions belong to L^p. So the function $g(x) \equiv K$ is in L^p. Now apply the proposition. □

One might suppose that L^p convergence implies almost everywhere convergence. But the next example shows that that is not the case.

Example: Let $X = [0,1]$, $\mathcal{B}$ be the Borel sets, and μ be Lebesgue measure. Consider the intervals in $[0,1]$ with dyadic[1] endpoints. Order these intervals in decreasing order of size. Let f_j be the characteristic function of the jth interval.

Then it is clear that the f_j tend to $f \equiv 0$ in L^p norm. But, if x is any point of $[0,1]$, then there is a subsequence f_{j_k} that equals 1 at x and there is another subsequence f_{j_ℓ} that equals 0 at x. So we do *not* have pointwise convergence at *any point* of the interval $[0,1]$.

[1] A point is dyadic if it has the form $j/2^k$.

In this section we treat a new concept of convergence which is analytically useful. And it is intuitively appealing.

Definition: Let $(X, \mathcal{X}, \mu)$ be a measure space. A sequence $\{f_j\}$ of measurable functions is said to *converge in measure* to a measurable function f precisely when

$$\lim_{j\to\infty} \mu(\{x \in \mathbb{R} : |f_j(x) - f(x)| \geq \alpha\}) = 0$$

for each $\alpha > 0$.

The sequence $\{f_j\}$ is said to be *Cauchy in measure* when

$$\lim_{j,k\to\infty} \mu(\{x \in \mathbb{R} : |f_j(x) - f_k(x)| \geq \alpha\}) = 0 \qquad (\star)$$

for each $\alpha > 0$.

Example: Let $f_j(x) = \chi_{[j,\infty)}$. Then the f_j do not converge in measure. Indeed they are not Cauchy in measure.

Let $g_j(x) = \chi_{[1,1+1/j]}$. Then the g_j converge in measure to the identically 0 function.

Proposition: *Let $(X, \mathcal{X}, \mu)$ be a measure space. If the functions $\{f_j\}$ converge in L^p, $1 \le p < \infty$, then the sequence converges in measure.*

Proof: Let $\alpha > 0$. Set

$$E_j(\alpha) = \{x \in \mathbb{R} : |f_j(x) - f(x)| \geq \alpha\} .$$

Then

$$\int |f_j - f|^p \, d\mu \geq \int_{E_j(\alpha)} |f_j - f|^p \, d\mu \geq \alpha^p \cdot \mu(E_j(\alpha)) .$$

We know that $\|f_j - f\|_{L^p} \to 0$. Since $\alpha > 0$, we may conclude that $\mu(E_j(\alpha)) \to 0$ as $j \to \infty$. $\square$