Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

The reader may note that an example from last time shows that a sequence of functions can converge in measure while not converging pointwise at any point. However the following result of F. Riesz tends to ameliorate the situation.

Proposition (Riesz): *Let $(X, \mathcal{X}, \mu)$ be a measure space. Suppose that $\{f_j\}$ is a sequence of measurable functions that is Cauchy in measure. Then there is a subsequence which converges almost everywhere and in measure to a measurable limit function f.*

Proof: Choose a subsequence $\{f_{j_k}\}$ so that $E_k = \{x \in \mathbb{R} : |f_{j_{k+1}}(x) - f_{j_k}(x)| \geq 2^{-k}\}$ satisfies $\mu(E_k) < 2^{-k}$. Set $F_k = \cup_{j=k}^{\infty} E_j$. Thus $F_k \in \mathcal{X}$ and $\mu(F_k) < 2^{-(k-1)}$.

If $\ell \geq m \geq n$ and $x \notin F_n$, then

$$
\begin{aligned}
|f_{j_\ell}(x) - f_{j_m}(x)| &\leq |f_{j_\ell}(x) - f_{j_{\ell-1}}(x)| + \cdots + |f_{j_{m+1}}(x) - f_{j_m}(x)| \\
&\leq \frac{1}{2^{\ell-1}} + \cdots + \frac{1}{2^m} \\
&< \frac{1}{2^{m-1}} \,. \qquad (*)
\end{aligned}
$$

Let $F = \cap_{n=1}^{\infty} F_n$, so that $F \in \mathcal{X}$ and $\mu(F) = 0$. From the above reasoning it follows that $\{f_{j_m}\}$ converges on $X \setminus F$. If we define

$$f(x) = \begin{cases} \lim_{m\to\infty} f_{j_m}(x) & \text{if} \quad x \notin F\,, \\ 0 & \text{if} \quad x \in F\,, \end{cases}$$

then $\{f_{j_m}\}$ converges almost everywhere to the measurable function f. Passing to the limit in $(*)$ as $\ell \to \infty$, we conclude that, if $m \geq n$ and $x \notin F_n$, then

$$|f(x) - f_{j_m}| \leq \frac{1}{2^{m-1}} \leq \frac{1}{2^{n-1}}\,.$$

This proves that the sequence $\{f_{j_m}\}$ converges uniformly to f on the complement of each set F_n.

To see that $\{f_{j_m}\}$ converges in measure to f, let α, ϵ be positive real numbers and choose n so large that

$$\mu(F_n) < 2^{-(n-1)} < \min(\alpha, \epsilon)\,.$$

If $m \geq n$, then the above estimate shows that

$$\begin{aligned} \{x \in \mathbb{R} : |f(x) - f_{j_m}(x)| \geq \alpha\} &\subseteq \{x \in \mathbb{R} : |f(x) - f_{j_m}(x)| > 2^{-(n-1)}\} \\ &\subseteq F_n\,. \end{aligned}$$

Thus

$$\mu\{x \in \mathbb{R} : |f(x) - f_{j_m}(x)| \geq \alpha\}) \leq \mu(F_n) < \epsilon$$

for all $m \geq n$. We conclude that $\{f_{j_m}\}$ converges in measure to f. $\square$

Corollary: *Let* $\{f_j\}$ *be a sequence of measurable functions which is Cauchy in measure. Then there is a measurable function* f *to which the sequence converges in measure. This limit function is uniquely determined almost everywhere.*

Proof: We know that there is a subsequence $\{f_{j_k}\}$ that converges in measure to f. To see that the entire sequence converges in measure to f, notice that

$$|f(x) - f_j(x)| \leq |f(x) - f_{j_k}(x)| + |f_{j_k}(x) - f_j(x)| \, .$$

Thus

$$\begin{aligned} \{x \in \mathbb{R} : |f(x) - f_j(x)| \geq \alpha\} \quad \subseteq \quad & \left\{x \in \mathbb{R} : |f(x) - f_{j_k}(x)| \geq \frac{\alpha}{2}\right\} \\ & \bigcup \left\{x \in \mathbb{R} : |f_{j_k}(x) - f_j(x)| \geq \frac{\alpha}{2}\right\} . \end{aligned}$$

The convergence in measure of $\{f_j\}$ follows from this relation.

Now suppose that the sequence $\{f_j\}$ converges in measure both to f and to g. Since

$$|f(x) - g(x)| \le |f(x) - f_j(x)| + |f_j(x) - g(x)| ,$$

it follow that

$$\begin{aligned} \{x \in \mathbb{R} : |f(x) - g(x)| \ge \alpha\} \quad \subseteq \quad & \left\{x \in \mathbb{R} : |f(x) - f_j(x)| \ge \frac{\alpha}{2}\right\} \\ & \bigcup \left\{x \in \mathbb{R} : |f_j(x) - g(x)| \ge \frac{\alpha}{2}\right\} , \end{aligned}$$

hence, passing to the limit,

$$\mu(\{x \in \mathbb{R} : |f(x) - g(x)| \ge \alpha\}) = 0$$

for all $\alpha > 0$. Taking $\alpha = 1/n$ for $n \in \mathbb{N}$, we conclude that $f = g$ a.e. □

Certainly convergence in L^p implies convergence in measure. The converse is not true, as the next example shows.

Example: Let μ be Lebesgue measure on the σ-algebra $\mathcal{B}$. Let $f_j(x) = j \cdot \chi_{[j,j+1/j]}$. Then f_j converges in measure to the identically 0 function, but the f_j do not converge in L^p norm.

However, with an additional hypothesis, we can obtain a positive result.

Proposition: *Let $\{f_j\}$ be a sequence of functions in L^p which converges in measure to a function f. Let $g \in L^p$ satisfy*

$$|f_j(x)| \leq g(x) \quad a.e.$$

Then $f \in L^p$ and the sequence $\{f_j\}$ converges to f in L^p.

Proof: If $\{f_j\}$ does not converge in L^p to f, then there exists a subsequence $\{f_{j_k}\}$ and an $\epsilon > 0$ such that

$$\|f_{j_k} - f\|_{L^p} > \epsilon \quad \text{for} \quad k \in \mathbb{N} . \qquad (**)$$

Since $\{f_{j_k}\}$ is a subsequence of $\{f_j\}$, we see that it converges in measure to f. By an earlier proposition, there is a subsequence $\{f_{j_{k_\ell}}\}$ which converges almost everywhere and in measure to a function h. From the uniqueness part of the corollary, it follows that $h = f$ a.e. Since $\{f_{j_{k_\ell}}\}$ converges almost everywhere to f and is dominated by g, our proposition implies that $\|f_{j_{k_\ell}} - f\|_{L^p} \to 0$. But this contradicts $(**)$. $\square$

Our concluding result for this lesson is quite striking, and ties together many of the ideas that we have introduced.

Theorem (Vitali): *Let $(X, \mathcal{X}, \mu)$ be a measure space. Let $\{f_j\}$ be a sequence in L^p, $1 \le p < \infty$. Then the following three conditions taken together are necessary and sufficient for the L^p convergence of $\{f_j\}$ to f.*

(i) *$\{f_j\}$ converges to f in measure.*

(ii) *For each $\epsilon > 0$, there is a set $E_\epsilon \in \mathcal{X}$ with $\mu(E_\epsilon) < +\infty$, such that, if $F \in \mathcal{X}$ and $F \cap E_\epsilon = \emptyset$, then*

$$\int_F |f_j|^p \, d\mu < \epsilon^p \quad \text{for all } \; j \in \mathbb{N} \, .$$

(iii) *For each $\epsilon > 0$, there is a $\delta > 0$ so that, if $E \in \mathcal{X}$ and $\mu(E) < \delta$, then*

$$\int_E |f_j|^p \, d\mu < \epsilon^p \quad \text{for all } \; j \in \mathbb{N} \, .$$

Proof: We know that L^p convergence implies convergence in measure. The fact that L^p convergence of the $\{f_j\}$ implies **(ii)** and implies **(iii)** is straightforward and we leave the matter as an exercise for the reader.

We shall next prove that the three given conditions taken together imply that $\{f_j\}$ converges in the L^p topology to f. Let $\epsilon > 0$. Let E_ϵ be as in statement **(ii)** and let $F = X \setminus E_\epsilon$. If the Minkowski inequality is applied to

$$f_j - f_k = [(f_j - f_k)\chi_{E_\epsilon}] + [f_j\chi_F - f_k\chi_F] \, ,$$

then we have

$$\|f_j - f_k\|_{L^p} \leq \left\{ \int_{E_\epsilon} |f_j - f_k|^p \, d\mu \right\}^{1/p} + 2\epsilon$$

for $j, k \in \mathbb{N}$.

Now let $\alpha = \epsilon[\mu(E_\epsilon)]^{-1/p}$. Let

$$H_{jk} = \{x \in E_\epsilon : |f_j(x) - f_k(x)| \geq \alpha\} .$$

Because of **(i)**, there is a number K such that, if $j, k \geq K$, then $\mu(H_{jk}) < \delta$. Another application of Minkowski together with **(iii)** gives

$$\begin{aligned} \left\{\int_{E_\epsilon} |f_j - f_k|^p \, d\mu\right\}^{1/p} &\leq \left\{\int_{E_\epsilon \setminus H_{jk}} |f_j - f_k|^p \, d\mu\right\}^{1/p} \\ &\quad + \left\{\int_{H_{jk}} |f_j|^p \, d\mu\right\}^{1/p} + \left\{\int_{H_{jk}} |f_k|^p \, d\mu\right\}^{1/p} \\ &\leq \alpha[\mu(E_\epsilon)]^{1/p} + \epsilon + \epsilon \\ &\leq 3\epsilon , \end{aligned}$$

for $j, k \geq K$.

Combining this last result with the first inequality in the proof, we find that the sequence $\{f_j\}$ is Cauchy in L^p and hence convergent in L^p. Since we already know that $\{f_j\}$ converges in measure to f, we may conclude from the uniqueness part of the corollary above that $\{f_j\}$ converges to f in L^p. □