Math 4121
April 2, 2021 Lecture

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March 27, 2021

Figure: This is your instructor.

The Lebesgue Integral

Higher dimensional real analysis is much more complex and fascinating than analysis on the real line. Thus we are certainly interested in doing measure theory in $\mathbb{R}^2$, for instance. We could, if we wished, use the open sets in $\mathbb{R}^2$ to generate the σ-algebra of Borel sets and proceed from there. But it is natural to think of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and to wonder whether the 1-dimensional Lebesgue measure on each of the $\mathbb{R}$ factors can somehow be combined to produce a product measure on $\mathbb{R}^2$.

It turns out that the answer is "yes," and in fact the two approaches described in the last paragraph turn out to be essentially equivalent. That is the subject of this discussion. Note that the big theorems on this topic are those of Tonelli and Fubini.

Definition: Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measure spaces. Then a set of the form $A \times B$, with $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ is called a *measurable rectangle*, or sometimes simply a *rectangle* in $Z \equiv X \times Y$. See the figure. We denote the collection of all finite unions of rectangles by $\mathcal{Z}$.

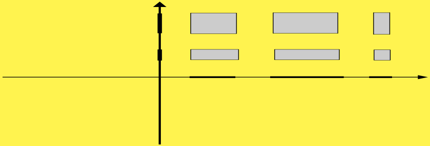

Figure: A measurable rectangle.

As an exercise, you should verify that every element of $\mathcal{Z}$ can be written as a finite pairwise *disjoint* union of rectangles.

Lemma: *The collection $\mathcal{Z}$ is an algebra of subsets of Z.*

Proof: Clearly the finite union of sets in $\mathcal{Z}$ is also in $\mathcal{Z}$. It is also straightforward to check that the complement of a rectangle in $\mathcal{Z}$ is (not necessarily itself a rectangle but) also an element of $\mathcal{Z}$. By de Morgan's law, we see that the complement of any set in $\mathcal{Z}$ belongs to $\mathcal{Z}$. $\square$

Definition: If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are measure spaces, then $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ denotes the σ-algebra of subsets of $Z = X \times Y$ generated by rectangles $A \times B$ with $A \in \mathcal{X}$ and $B \in \mathcal{Y}$. A set in $\mathcal{Z}$ is called a *$\mathcal{Z}$-measurable set*, or sometimes just a *measurable subset of Z*.

Theorem (Product Measure Theorem): *Let* $(X, \mathcal{X}, \mu)$ *and* $(Y, \mathcal{Y}, \nu)$ *be measure spaces. Then there is a measure* π *defined on* $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ *such that*

$$\pi(A \times B) = \mu(A) \cdot \nu(B)$$

for all $A \in \mathcal{X}$ *and* $B \in \mathcal{Y}$*. If the two measure spaces are* σ*-finite, then there is a unique measure* π *with the displayed property.*

Proof: Suppose that the set $A \times B$ is the pairwise disjoint union of a collection $\{A_j \times B_j\}$ of rectangles, $j = 1, 2, \ldots$. Therefore

$$\chi_{A \times B}(x, y) = \chi_A(x) \cdot \chi_B(y) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \cdot \chi_{B_j}(y)$$

for all $x \in X$, $y \in Y$. Holding x fixed, we integrate in y with respect to ν. Applying the monotone convergence theorem then yields

$$\chi_A(x) \cdot \nu(B) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \cdot \nu(B_j)\,.$$

A second application of the monotone convergence theorem then yields that

$$\mu(A) \cdot \nu(B) = \sum_{j=1}^{\infty} \mu(A_j) \cdot \nu(B_j) \, .$$

Let $E \in \mathcal{Z}$. We may assume that

$$E = \bigcup_{j=1}^{\infty} (A_j \times B_j) \; ,$$

where the sets $A_j \times B_j$ are pairwise disjoint rectangles. If we define

$$\pi(E) = \sum_{j=1}^{\infty} \mu(A_j) \cdot \nu(B_j) \, ,$$

then the argument above implies that π is well defined and countably additive on $\mathcal{Z}$. By the Carathéodory extension theorem, there is an extension of π to a measure $\widehat{\pi}$ on the σ-algebra $\widehat{\mathcal{Z}}$ generated by $\mathcal{Z}$. Since $\widehat{\pi}$ is an extension of π, it is clear that the identity in the statement of the theorem holds.

In the case that $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ are σ-finite, then π is a σ-finite measure on the algebra $\widehat{\mathcal{Z}}$ and the uniqueness of a measure satisfying the displayed equation follows from the uniqueness assertion of the Hahn extension theorem. $\square$

The measure constructed in the above theorem is called the *product* of μ and ν. In the case that μ and ν are both σ-finite, then they have a unique product.

Our next goal is, as we predicted in the introductory paragraph, to relate integration with respect to a product measure to iterated integration.

Definition: Let E be a subset of $Z = X \times Y$. Let $x \in X$. Then the *x-section* of E is the set

$$E_x = \{y \in Y : (x, y) \in E\} .$$

Similarly, if $y \in Y$, then the *y-section* of E is the set

$$E^y = \{x \in X : (x, y) \in E\} .$$

Definition: If $f : Z \to \widehat{\mathbb{R}}$ and if $x \in X$, then the *x-section* of f is the function f_x defined on Y by

$$f_x(y) = f(x,y) \; , \quad y \in Y \, .$$

Similarly, if $y \in Y$, then the *y-section* of f is the function f^y defined on X by

$$f^y(x) = f(x,y) \; , \quad x \in X \, .$$

Lemma:

(a) *If E is a measurable subset of Z, then every section of E is measurable.*

(b) *If $f : Z \to \widehat{\mathbb{R}}$ is a measurable function, then every section of f is a measurable function.*

Proof:

(a) If $E = A \times B$ and $x \in X$, then the x-section of E is B if $x \in A$ and is $\emptyset$ if $x \notin A$. Thus the rectangles are contained in the collection $\mathcal{E}$ of all sets in $\mathcal{Z}$ having the property that each x-section is measurable. It is clear that $\mathcal{E}$ is a σ-algebra. Hence it follows that $\mathcal{E} = \mathcal{Z}$.

(b) Let $x \in X$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \{y \in Y : f_x(y) > \alpha\} &= \{y \in Y : f(x,y) > \alpha\} \\ &= \{(x,y) \in X \times Y : f(x,y) > \alpha\}_x . \end{aligned}$$

Thus if f is $\mathcal{Z}$-measurable, then f_x is $\mathcal{Y}$-measurable. Similarly, f^y is $\mathcal{X}$-measurable. □

Recall that a monotone class is a nonempty collection $\mathcal{M}$ of sets which contains the union of each increasing sequence in $\mathcal{M}$ and also the intersection of each decreasing sequence in $\mathcal{M}$. It is straightforward to check that if $\mathcal{A}$ is a nonempty collection of subsets of a set S, then the σ-algebra $\mathcal{S}$ generated by $\mathcal{A}$ contains the monotone class $\mathcal{M}$ generated by $\mathcal{A}$. Next we show that, if $\mathcal{A}$ is an algebra, then $\mathcal{S} = \mathcal{M}$.

Lemma (Monotone Class Lemma): *If $\mathcal{A}$ is an algebra of sets, then the σ-algebra $\mathcal{S}$ generated by $\mathcal{A}$ coincides with the monotone class $\mathcal{M}$ generated by $\mathcal{A}$.*

Proof: Clearly $\mathcal{M} \subseteq \mathcal{S}$. To prove the reverse inequality it is sufficient (because $\mathcal{M}$ is also a monotone class) to prove that $\mathcal{M}$ is an algebra.

Now let $E \in \mathcal{M}$. Define $\mathcal{M}(E)$ to be the collection of sets $F \in \mathcal{M}$ so that $E \setminus F$, $E \cap F$, and $F \setminus E$ all belong to $\mathcal{M}$. Clearly $\emptyset, E \in \mathcal{M}(E)$. Also $\mathcal{M}(E)$ is obviously a monotone class. In addition, $F \in \mathcal{M}(E)$ if and only if $E \in \mathcal{M}(F)$.

If the set E belongs to the algebra $\mathcal{A}$, then clearly $\mathcal{A} \subseteq \mathcal{M}(E)$. But, since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$, we must have $\mathcal{M}(E) = \mathcal{M}$ for $E \in \mathcal{A}$. Thus, if $E \in \mathcal{A}$ and $F \in \mathcal{M}$, then $F \in \mathcal{M}(E)$. We conclude then that, if $E \in \mathcal{A}$ and $F \in \mathcal{M}$, then $E \in \mathcal{M}(F)$ and therefore $\mathcal{A} \subseteq \mathcal{M}(F)$ for any $F \in \mathcal{M}$. The minimality of $\mathcal{M}$ now implies that $\mathcal{M}(F) = \mathcal{M}$ for any $F \in \mathcal{M}$. So $\mathcal{M}$ is closed under intersection and relative complements. Since $X \in \mathcal{M}$, it is now plain that $\mathcal{M}$ is an algebra. Since $\mathcal{M}$ is also a monotone class, it is in fact a σ-algebra. $\square$