# Math 4121
# April 5, 2021 Lecture

Steven G. Krantz

March 27, 2021

Figure: This is your instructor.

# The Lebesgue Integral

# Measure on a Product Space

The monotone class lemma tells us in particular that, if a monotone class contains an algebra $\mathcal{A}$, then it contains the $\sigma$-algebra generated by $\mathcal{A}$.

**Lemma:** *Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be $\sigma$-finite measure spaces. If $E \in \mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, then the functions defined by*

$$f(x) = \nu(E_x) \quad \textit{and} \quad g(y) = \mu(E^y)$$

*are measurable, and*

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\nu \, .$$

**Proof:** Suppose at first that the two measure spaces are finite. Let $\mathcal{M}$ be the collection of all $E \in \mathcal{Z}$ for which the first displayed equations are true. We shall prove that $\mathcal{M} = \mathcal{Z}$ by showing that $\mathcal{M}$ is a monotone class containing the algebra $\mathcal{Z}$.

In point of fact, if $E = A \times B$ with $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, then

$$f(x) = \chi_A(s) \cdot \nu(B) \quad , \quad g(y) = \chi_B(y) \cdot \mu(A) \, ,$$

are measurable and furthermore

$$\int_X f \, d\mu = \mu(A) \cdot \nu(B) = \int_Y g \, d\nu \, .$$

Since any element of $\mathcal{Z}$ can be written as a finite disjoint union of rectangles, we may conclude that $\mathcal{Z} \subseteq \mathcal{M}$.

Next we show that $\mathcal{M}$ is a monotone class. In fact, let $\{E_j\}$ be a monotone increasing sequence in $\mathcal{M}$ with union $E$. Then

$$f_j(x) = \nu((E_j)_x) \quad , \quad g_j(y) = \mu((E_j)^y)$$

are both measurable functions and

$$\int_X f_j \, d\mu = \pi(E_j) = \int_Y g_j \, d\nu \, .$$

Clearly the monotone increasing sequences $\{f_j\}$ and $\{g_j\}$ converge to the functions $f$ and $g$ defined by

$$f(x) = \nu(E_x) \quad \text{and} \quad g(y) = \mu(E^y)$$

respectively. If we use the fact that $\pi$ is a measure together with the monotone convergence theorem, we may conclude that

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\nu \, .$$

Hence $E \in \mathcal{M}$.

Since $\pi$ is a finite measure, one can show in just the same way that if $\{F_j\}$ is a monotone decreasing sequence in $\mathcal{M}$, then $F = \cap_j F_j$ belongs to $\mathcal{M}$. Thus $\mathcal{M}$ is a monotone class, and it follows from the monotone class lemma that $\mathcal{M} = \mathcal{Z}$.

We refer the reader to Bartle's book for the details of the $\sigma$-finite case. $\square$

**Theorem (Tonelli's Theorem):** *Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be $\sigma$-finite measure spaces. Let $Z = X \times Y$ and let $\psi : Z \to \widehat{\mathbb{R}}$ be measurable and nonnegative. Then the functions defined on $X$ and $Y$ by*

$$f(x) = \int_Y \psi_x \, d\nu \quad \text{and} \quad g(y) = \int_X \psi^y \, d\mu$$

*are measurable and*

$$\int_X f \, d\mu = \int_Z \psi \, d\pi = \int_Y g \, d\nu \, .$$

*In other words,*

$$\int_X \left( \int_Y \psi \, d\nu \right) d\mu = \int_Z \psi \, d\pi = \int_Y \left( \int_X \psi \, d\mu \right) d\nu \, .$$

**Proof:** If $\psi$ is the characteristic function of a set in $Z$, then the conclusion of the theorem is immediate from the lemma. By linearity, the theorem certainly holds for a measurable, simple function.

If now $\psi : Z \to \widehat{\mathbb{R}}$ is an arbitrary nonnegative, measurable function, then we know that there is a sequence $\{s_j\}$ of nonnegative, measurable simple functions which converge monotonically on $Z$ to $\psi$. If $\varphi_j$, $\psi_j$ are defined by

$$\varphi_j(x) = \int_Y (s_j)_x \, d\nu \quad \text{and} \quad \psi_j(x) = \int_X (s_j)^y \, d\mu \, ,$$

then $\varphi_j$, $\psi_j$ are measurable and monotone in the index $j$. By the monotone convergence theorem, $\{\varphi_j\}$ converges on $X$ to $f$ and $\{\psi_j\}$ converges on $Y$ to $g$.

Yet another application of the monotone convergence theorem yields

$$\begin{aligned}\int_X f\, d\mu &= \lim_{j\to\infty}\int_X \varphi_j\, d\mu \\ &= \lim_{j\to\infty}\int_Z s_j\, d\pi \\ &= \lim_{j\to\infty}\int_Y \psi_j\, d\nu \\ &= \int_Y g\, d\nu\,.\end{aligned}$$

In the same way one can show that

$$\int_Z \psi d\pi = \lim_{j\to\infty}\int_Z s_j d\pi\,.$$

From this we may conclude the second displayed equation in the theorem. □

As we can see, Tonelli's theorem answers the main question posed at the start of this chapter about realizing integrals on product spaces in two different ways—but only for nonnegative functions. It was Fubini who realized how to treat the case of functions that take both positive and negative values.

**Theorem (Fubini's Theorem):** *Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be $\sigma$-finite measure spaces. Let the measure $\pi$ on $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ be the product of $\mu$ and $\nu$. Let $Z = X \times Y$. If the function $\psi : Z \to \mathbb{R}$ is integrable with respect to $\pi$, then the extended real-valued functions defined almost everywhere by*

$$f(x) = \int_Y \psi_x \, d\nu \quad \text{and} \quad g(y) = \int_X \psi^y \, d\mu$$

*have finite integrals and*

$$\int_X f \, d\mu = \int_Z \psi \, d\pi = \int_Y g \, d\nu \, .$$

*In other words,*

$$\int_X \left( \int_Y \psi \, d\nu \right) d\mu = \int_Z \psi \, d\pi = \int_Y \left( \int_X \psi \, d\mu \right) d\nu \, .$$

**Proof:** Since $\psi$ is integrable with respect to $\pi$, then both its positive and negative parts $\psi^+$ and $\psi^-$ are integrable. We apply Tonelli's theorem to $\psi^+$ and to $\psi^-$ to conclude that the corresponding $f^+$ and $f^-$ have finite integrals with respect to $\mu$. Therefore $f^+$ and $f^-$ are finite-valued $\mu$-almost everywhere, so their difference $f$ is defined $\mu$-almost everywhere and the first part of the second displayed equation is clear. The second part of the second displayed equation is proved in a similar fashion. $\square$

It is worth noting that the functions defined in the first displayed equation are equal almost everywhere to integrable functions. The most important hypothesis in Fubini's theorem is that $\psi$ be integrable.