Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Additivity for Outer Measure

This chapter presents the striking fact that the outer measure is additive over the union of two disjoint sets provided that only *one of them* is measurable. We will also establish some other additivity and non-additivity properties of m^*.

Theorem: *Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let F be any subset of $\mathbb{R}$. Then*

(a) $m^*(E \cup F) + m^*(E \cap F) = m(E) + m^*(F)$.

(b) *If* $E \cap F = \emptyset$, *then* $m^*(E \cup F) = m(E) + m^*(F)$.

(c) *If* $m(E) < \infty$ *and if* $E \subseteq F$, *then we have* $m^*(F \setminus E) = m^*(F) - m(E)$.

Proof: Since $E \in \mathcal{L}$, it follows from Carathéodory's condition that $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$ for any set $A \subseteq \mathbb{R}$. In fact let $A = E \cup F$. Then

$$\begin{aligned} m^*(E \cup F) &= m^*((E \cup F) \cap E) + m^*((E \cup F) \setminus E) \\ &= m(E) + m^*(F \setminus E) . \end{aligned}$$

If we take $A = F$, then we have

$$m^*(F) = m^*(F \cap E) + m^*(F \setminus E) .$$

Thus we have

$$\begin{aligned} m^*(E \cup F) + m^*(E \cap F) &= [m(E) + m^*(F \setminus E)] + m^*(E \cap F) \\ &= m(E) + [m^*(F \setminus E) + m^*(E \cap F)] \\ &= m(E) + m^*(F) \,. \end{aligned}$$

This establishes **(a)**.

For **(b)**, if $E \cap F = \emptyset$, then $m^*(E \cap F) = 0$, thus the desired conclusion is immediate.

For **(c)**, let $G = F \setminus E$. Hence $F = E \cup G$ and $E \cap G = \emptyset$. From **(b)** we then conclude that

$$\begin{aligned} m^*(F) &= m^*(E \cup G) \\ &= m(E) + m^*(G) \\ &= m(E) + m^*(F \setminus E) \,. \end{aligned}$$

Since $m(E) < \infty$, we have that $m^*(F)$ and $m^*(F \setminus E)$ are either both $+\infty$ or both finite. Thus **(c)** follows. □

We saw earlier that, for any set $E \subseteq \mathbb{R}$, there is a G_δ set H so that $E \subseteq H$ and $m^*(E) = m(H)$. Furthermore, E is Lebesgue measurable if and only if $H \setminus E$ is a null set. The next result is in the same vein, but from the perspective of “within.”

Theorem: *If $m^*(E) < \infty$, then E is measurable if and only if there is a measurable set $W \subseteq E$ with $m(W) = m^*(E)$.*

Proof: If E is measurable then clearly we can just take $W = E$.

For the converse, suppose that $W \in \mathcal{L}$, $W \subseteq E$, and $m(W) = m^*(E)$. It then follows from part **(c)** of the theorem above that

$$m^*(E \setminus W) = m^*(E) - m(W) = 0 \,.$$

As a result, the null set $E \setminus W$ is Lebesgue measurable and hence $E = (E \setminus W) \cup W$ is also measurable. □

Now we present a refined and, in effect, simpler version of Carathéodory's criterion.

Theorem: *Let $A \subseteq \mathbb{R}$ be Lebesgue measurable with $m(A) < \infty$. Then $E \subseteq A$ is Lebesgue measurable if and only if*

$$m(A) = m^*(E) + m^*(A \setminus E) . \qquad (*)$$

Proof: If E is measurable, then the claimed result is immediate from Carathéodory's condition.

Conversely, by the theorem applied to $A \setminus E$, there is a G_δ set W with $A \setminus E \subseteq W$ and $m^*(A \setminus E) = m(W)$. Since $A \setminus E \subseteq A \cap W \subseteq W$, we may infer that

$$m^*(A \setminus E) \le m(A \cap W) \le m(W) = m^*(A \setminus E)\,.$$

Thus we conclude that $m(A \cap W) = m^*(A \setminus E)$. But $A \cap W$ is measurable. Also

$$A \cap (A \cap W) = A \cap W \quad \text{and} \quad A \setminus (A \cap W) = A \setminus W\,,$$

so we conclude that

$$m(A) = m(A \cap W) + m(A \setminus W) = m^*(A \setminus E) + m(A \setminus W)\,.$$

Using now equation $(*)$, we find that

$$m(A \setminus W) = m^*(E) .$$

Let $B = A \setminus W \subseteq E$. It follows from the usual theorem that E is Lebesgue measurable. $\square$

Often we are studying sets which all lie in a large interval of the form $I_n = [-n, n]$ for n a large positive integer. The following result is then useful.

Corollary: *A set $E \subseteq I_n$ is Lebesgue measurable if and only if*

$$m(I_n) = m^*(E) + m^*(I_n \setminus E)\,.$$

Proof: This result is immediate from the theorem and the fact that I_n is measurable. □

When we are studying unbounded sets, the following result is often useful.

Theorem: *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if and only if the sets $E \cap I_n$ are measurable for each $n \in \mathbb{N}$.*

Proof: If E is measurable then the result is obvious.

Conversely, if each $E_n = E \cap I_n$ is measurable for each n, then it follow from the fact that $E = \cup_{n=1}^{\infty} E_n$ that E is measurable. □

The most obvious notion of inner measure would be to approximate a measurable set from the inside by intervals. But this is unrealistic because there are many sets of positive measure that contain no intervals.

An alternative approach is this. Suppose that $E \subseteq I_n$ for some n. Define the *inner measure* $m_*(E)$ of E to be

$$m_*(E) = m(I_n) - m^*(I_n \setminus E) .$$

Now our corollary takes this form:

Corollary: *A set $E \subseteq I_n$ is Lebesgue measurable if and only if its inner measure and its outer measure are equal.*

This last result is useful, for instance, if one is studying sets which are all subsets of a fixed bounded interval.