Math 4121
April 9, 2021 Lecture

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April 3, 2021

Figure: This is your instructor.

The Lebesgue Integral

Nonmeasurable Sets and Non-Borel Sets

In this penultimate lecture we talk about nonmeasurable sets and non-Borel sets. Some interesting examples are generated along the way.

Definition: If $A \subseteq \mathbb{R}$, then its *difference set* is defined to be

$$A \ominus A = \{a - b : a \in A, b \in A\} .$$

Note that, if $A \subseteq B$, then $A \ominus A \subseteq B \ominus B$.

Lemma: *Let $K \subseteq \mathbb{R}$ be a compact set with $m(K) > 0$. Then the difference set $K \ominus K$ contains an open interval with center at the origin of coordinates.*

Proof: Since $0 < m(K) < \infty$, there exists an open set U with $K \subseteq U$ and $m(U) < 2m(K)$. Because K is compact and ${}^cU = \mathbb{R} \setminus U$ is closed, we have that

$$\delta = \operatorname{dist}(K, {}^cU) > 0 \,.$$

Thus we have that if $|x| = \operatorname{dist}(x, 0) < \delta$, then $x + K \subseteq U$.

We claim that $(x + K) \cap K \neq \emptyset$. If not, then since $K \cup (x + K) \subseteq U$ and $(x + K) \cap K = \emptyset$ and m is additive,

$$\begin{aligned} 2m(K) &= m(K) + m(x + K) \\ &= m(K \cup (x + K)) \\ &\leq m(U) < 2m(K) \,. \end{aligned}$$

That is a contradiction.

We conclude then that $(x + K) \cap K \neq \emptyset$ for all x with $|x| < \delta$. But then we have that, if $|x| < \delta$, then there exist $k_1, k_2 \in K$ such that $x = k_1 - k_2 \in K \ominus K$. Thus the set $K \ominus K$ contains the open ball with center 0 and radius δ. $\square$

Theorem: *Let $E \subseteq \mathbb{R}$ be any Lebesgue measurable set having positive measure. Then the difference set $E \ominus E$ contains an open interval centered at 0.*

Proof: For $n \in \mathbb{N}$, let $E_n = \{x \in E : |x| < n\}$. Since $m(E) = \lim_{n\to\infty} m(E_n)$, we see that $m(E_n) > 0$ for n sufficiently large—say that $n \geq n_0$. Certainly $0 < m(E_{n_0}) < \infty$. Thus, by a theorem above, there exists a compact set $K \subseteq E_{n_0} \subseteq E$ such that

$$0 < \frac{1}{2} \cdot m(E_{n_0}) \leq m(K) .$$

Since $K \subseteq E$, we know that $K \ominus K \subseteq E \ominus E$. The preceding lemma now tells us that $K \ominus K$ contains an open interval with center 0. Hence so does $E \ominus E$. □

Theorem: *A set $E \subseteq \mathbb{R}$ with positive outer measure contains a nonmeasurable subset.*

Proof: Review the argument presented early in the course which established the existence of a nonmeasurable set. We called that nonmeasurable set S, and we considered its translates $S_q = S + q = \{s + q : s \in S\}$ for each rational q. Of course each S_q is nonmeasurable, just because it is a translate of S. But it is conceivable that, for some q, $E_q \equiv S_q \cap E$ *is* measurable.

However, if E_q is measurable for some q and has positive measure, then the preceding theorem tells us that the difference set $E_q \ominus E_q$ must contain a nontrivial open interval. Since $E_q \subseteq S_q$, it follows that $S_q \ominus S_q$ also must contain a nontrivial open interval. That contradicts the construction of S, just because $S \ominus S$ cannot contain any rationals except 0 (and the same assertion holds for $S_q \ominus S_q$). We conclude that those sets E_q which are measurable must be null sets.

Again referring to our much earlier arguments, we have that

$$E = \bigcup_{q \in \mathbb{Q}} E \cap S_q = \bigcup_{q \in \mathbb{Q}} E_q \,.$$

If all of the sets E_q are measurable, then we know that they must be null sets. Hence E is a null set, and that is a contradiction. Thus at least one of the sets E_q is nonmeasurable. □

Now we strengthen this result as follows.

Theorem: *Let E be a Lebesgue measurable set such that $0 < m(E) < \infty$. Then there exist nonmeasurable subsets S and T of E such that $E = S \cup T$, $S \cap T = \emptyset$, and*

$$m(E) < m^*(S) + m^*(T) .$$

Proof: The preceding theorem tells us that the set E has a nonmeasurable subset S. Set $T = E \setminus S$. Thus $E = S \cup T$ and $S \cap T = \emptyset$. Furthermore, since $S = E \setminus T$, the set T must also be nonmeasurable.

The subadditivity of m^* now tells us that

$$m^*(E) \leq m^*(S) + m^*(T) .$$

If equality were to hold here, then we would know that S is a measurable set, which is false. Thus we have the asserted inequality in the theorem. □

Next we shall establish that every nonmeasurable set with finite outer measure is part of a nonadditive decomposition of a measurable set.

Theorem: *Let S be a nonmeasurable set so that $m^*(S) < \infty$. Let H be a G_δ set with $S \subseteq H$ and $m^*(S) = m(H)$. Then $T = H \setminus S$ is also nonmeasurable and*

$$m(H) = m(S \cup T) < m^*(S) + m^*(T) .$$

Proof: The existence of H was established earlier. Since $S = H \setminus T$, we see that T must be nonmeasurable. Thus $m^*(T) > 0$. The strict inequality then follows. □

Here we present a construction from Bartle's book of a Lebesgue measurable set that is not Borel. As part of this argument, we use the famous Cantor-Lebesgue function which you probably learned about in your undergraduate analysis course. We review that function here.

Let $\mathcal{C}$ be the Cantor ternary set. Let I denote the unit interval $[0,1]$. Each element x of the Cantor set has a ternary (instead of a decimal) expansion which can be written as

$$x = 0.c_1 c_2 c_3 \cdots ,$$

where each c_j is either a 0 or a 2. Let $I = [0,1]$ be the unit interval. We define a mapping $\varphi : \mathcal{C} \to I$ by

$$\varphi(x) = 0.(c_1/2)(c_2/2)(c_3/2) \cdots .$$

Here the number on the right should be interpreted as a binary expansion (all the digits are 0s or 1s).

Clearly, if $x, x^* \in \mathcal{C}$ and $x < x^*$, then there is a $j \in \mathbb{N}$ such that all the digits in the ternary expansion of x equal the corresponding digits in the ternary expansion of x^* up to the jth digit, but the jth digit of x is 0 while the jth digit of x^* is 2. It follows then that $\varphi(x) \le \varphi(x^*)$ so that φ is a monotone nondecreasing function from $\mathcal{C}$ to I.

We should note in passing that φ is *not* one-to-one. For example, if $x = 0.020\underline{2}$ and $x^* = 0.022\underline{0}$ then certainly $x < x^*$. [Here we underline a digit if it is repeated infinitely often.] Yet $\varphi(x) = 0.010\underline{1} = 0.011\underline{0} = \varphi(x^*)$. In fact the reader may check that $\varphi(x) = \varphi(x^*)$ precisely when x, x^* are the left and right endpoints of one of the removed ternary intervals in the construction of the Cantor set.

It should be noted, however, that φ does map C *onto* I. This is so because if $y = 0.b_1 b_2 b_3 \cdots$ is the binary expansion of any point y in I, then y is the image under φ of the point x which has ternary expansion $0.(2b_1)(2b_2)(2b_3)\cdots$.

Next we extend φ to be defined on all of I by defining it to be constant on each of the removed ternary intervals. In fact φ takes the same value at both endpoints of such a removed ternary interval, so that tells us what the constant value should be on that interval. As an instance,

$$\varphi(x) = 0.0\underline{1} = 0.1\underline{0} = \frac{1}{2}$$

for all x satisfying $0.0\underline{2} = 1/3 \leq x \leq 2/3 = 0.20$ in ternary.

This extended function, which we continue to denote by φ, is plainly a monotone nondecreasing function mapping I into I. It does not have any jump discontinuities (which are the only kind of discontinuities that a monotone function can have) since every value in I is taken at least once. So the extended function φ is continuous at every point of I.

It is worth noting that the derivative φ' equals 0 at each point of $I \setminus \mathcal{C}$, since φ is constant on a neighborhood of such a point. The extended function φ is usually called the *Cantor-Lebesgue function*. It is worth noting explicitly that $\varphi'(x) = 0$ for almost every $x \in I$.

Now we define

$$\psi(x) = \varphi(x) + x \, .$$

This new function ψ is a strictly increasing function from I to the closed interval $[0, 2]$. So it is one-to-one. And it is onto. It is also continuous. Hence the inverse function from $[0, 2]$ to I is also continuous. In other words, ψ is a homeomorphism from I to $[0, 2]$. It follows that both ψ and ψ^{-1} take Borel sets to Borel sets.

Since φ is constant on each of the ternary intervals that was removed in the construction of the Cantor set, we see that ψ maps each such interval into an interval of the same length. Thus

$$m(\psi(I \setminus \mathcal{C})) = m(I \setminus \mathcal{C}) = 1 \,.$$

Since $m([0,2]) = 2$ and $[0,2] = \psi(\mathcal{C}) \cup \psi(I \setminus \mathcal{C})$ and $\psi(\mathcal{C}) \cap \psi(I \setminus \mathcal{C}) = \emptyset$, we see that

$$2 = m(\psi(\mathcal{C})) + m(\psi(I \setminus \mathcal{C})) \,.$$

As a result, $m(\psi(\mathcal{C})) = 1$. In conclusion, the homeomorphism ψ maps the set $\mathcal{C}$, which has Lebesgue measure 0, to a set with Lebesgue measure 1.

Since $\psi(\mathcal{C})$ has positive measure, we know from an earlier theorem that it contains a set V that is *not* Lebesgue measurable. Then the set $V^* = \psi^{-1}(V)$ is a subset of $\mathcal{C}$ and hence is a Lebesgue null set. Therefore V^* *is* Lebesgue measurable. However V^* cannot be a Borel set; if it were, then $V = \psi(V^*)$ would also be Borel, and hence Lebesgue measurable. But this contradicts the choice of V as a nonmeasurable set.

We summarize the point of our analysis with an enunciated theorem.

Theorem: *There exists a Lebesgue measurable subset of* $\mathbb{R}$ *that is not Borel.*

One upshot of the proof that we just presented is that it is possible for a homeomorphism to map a Lebesgue measurable set to a nonmeasurable set.