Math 4121
April 14, 2021 Lecture

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April 4, 2021

Figure: This is your instructor.

The Lebesgue Integral

Applications

We take this opportunity to provide some elegant applications of Lebesgue measure theory to basic harmonic analysis. This will give the reader a glimpse of how useful Lebesgue measure can be.

Definition: Let f be an integrable function on $\mathbb{R}$. Define the *Hardy-Littlewood maximal function Mf* of f to be

$$Mf(x) = \sup_{R>0} \frac{1}{2R} \int_{x-R}^{x+R} |f(t)| \, d\mu(t) \, .$$

In the twentieth century, the Hardy-Littlewood maximal function became a fundamental tool of analysis. We shall learn two nice uses of this concept.

Definition: Let f be a measurable function on $\mathbb{R}$. We say that f is *weak type 1* if there is a constant $C > 0$ so that, for each $\lambda > 0$,

$$\mu\{x \in \mathbb{R} : |f(x)| > \lambda\} \leq \frac{C}{\lambda}\,.$$

Example: The function $f(x) = 1/x$ is weak type 1 on $\mathbb{R}$. It is *not*, however, Lebesgue integrable.

Definition: Let T be a linear operator on the space $L^1(\mathbb{R})$ of Lebesgue integrable functions, taking values in the measurable functions. We say that T is of *weak type* $(1, 1)$ if there is a positive constant C so that, for each $f \in L^1$ and each $\lambda > 0$,

$$\mu\{x \in \mathbb{R} : |Tf(x)| > \lambda\} \leq \frac{C\|f\|_{L^1}}{\lambda} .$$

Theorem: *The Hardy-Littlewood maximal operator M is weak type* $(1, 1)$.

In order to prove this result we need a geometric result that is commonly known as a *covering lemma*.

Lemma: *Let $K \subseteq \mathbb{R}$ be a compact set. Let $\{I_j\}$ be a finite collection of open intervals that covers K in the sense that $K \subseteq \cup_j I_j$. Write $I_j = (c_j - r_j, c_j + r_j)$, each j. Then there is a pairwise disjoint subcollection $\{I_{j_k}\}$ with the property that the three-fold dilates $3I_j = (c_j - 3r_j, c_j + 3r_j)$ cover K.*

Proof: Choose I_{j_1} which has the greatest length. If there is more than one then simply select one. Next choose I_{j_2} which is disjoint from I_{j_1} and which has greatest length. If there is more than one then simply select one. Continue in this fashion until the process stops—and it must stop because the collection $\{I_j\}$ is finite.

Now the chosen $\{I_{j_k}\}$ is a pairwise disjoint collection and we claim that $\{3I_{j_k}\}$ covers K. It suffices to show that $\{3I_{j_k}\}$ covers each of the original intervals I_j.

So fix an I_j. Let I_{j_k} be the first of the selected intervals that intersects I_j. Then the length of I_{j_k} must be greater than or equal to that of I_j by the way that we selected the I_{j_k}. But then it follows from the triangle inequality that $3I_{j_k}$ contains I_j. And that is what we wished to prove. $\square$

Proof of the Theorem: Fix $f \in L^1(\mathbb{R})$. Fix $\lambda > 0$. Let

$$S_\lambda = \{x \in \mathbb{R} : Mf(x) > \lambda\} .$$

Let K be a compact subset of S_λ. We shall prove that

$$\mu(K) \leq C \frac{\|f\|_{L^1}}{\lambda} .$$

The inner regularity of the Lebesgue measure will then yield the desired result.

Note that, for each $k \in K$, there is an interval $(k - \epsilon_k, k + \epsilon_k)$ so that

$$\frac{1}{2\epsilon_k} \int_{k-\epsilon_k}^{k+\epsilon_k} |f(t)| \, d\mu(t) > \lambda \, .$$

Now the intervals $I_k = (k - \epsilon_k, k + \epsilon_k)$ cover K. We may use the compactness of K to pass to a finite subcover. Then we may invoke the covering lemma to find a pairwise disjoint subcollection $\{I_{k_\ell}\}_{\ell=1}^m$ whose threefold dilates still cover K. Then we have

$$
\begin{aligned}
\mu(K) &\leq \mu\left(\bigcup_{\ell=1}^{m} 3I_{k_\ell}\right) \\
&\leq \sum_{\ell=1}^{m} \mu(3I_{k_\ell}) \\
&= \sum_{\ell=1}^{m} 3\mu(I_{k_\ell}) \\
&\leq 3\sum_{\ell=1}^{m} \frac{1}{\lambda}\int_{I_{k_\ell}} |f(t)|\, d\mu(t) \\
&\leq \frac{3}{\lambda}\cdot \|f\|_{L^1} .
\end{aligned}
$$

In the last inequality we have of course used the fact that the I_{k_ℓ} are pairwise disjoint.

That proves the result. $\square$

Now we formulate and prove a version of the celebrated Lebesgue differentiation theorem.

Theorem: *Let f be a Lebesgue integrable function on $\mathbb{R}$. Then, for almost every $x \in \mathbb{R}$, we have that*

$$\lim_{R \to 0} \frac{1}{2R} \int_{-R}^{R} f(t+x)\, dt = f(x)\,.$$

Proof: Fix $f \in L^1(\mathbb{R})$. Let $\epsilon > 0$. Choose a continuous, compactly supported function g so that $\|f - g\|_{L^1} < \epsilon^2$. Then we have

$$\begin{aligned}
&\mu\left\{x \in \mathbb{R} : \left|\limsup_{R\to 0} \frac{1}{2R}\int_{-R}^{R} f(t)\,dt - \liminf_{R\to 0} \frac{1}{2R}\int_{-R}^{R} f(t)\,dt\right| > \epsilon\right\} \\
&\leq \quad \mu\{x \in \mathbb{R} : \limsup_{R\to 0} \frac{1}{2R}\int_{-R}^{R} |f(t) - g(t)|\,dt > \epsilon/3\} \\
&\qquad + \mu\{x \in \mathbb{R} : \left|\limsup_{R\to 0} \frac{1}{2R}\int_{-R}^{R} g(t)\,dt \right. \\
&\qquad\qquad \left. - \liminf_{R\to 0} \frac{1}{2R}\int_{-R}^{R} g(t)\,dt\right| > \epsilon/3\} \\
&\qquad + \mu\{x \in \mathbb{R} : \limsup_{R\to 0} \frac{1}{2R}\int_{-R}^{R} |g(t) - f(t)|\,dt > \epsilon/3\}\,.
\end{aligned}$$

Now the middle term on the right is obviously 0 by the continuity of g. The first and last terms can be majorized as follows:

$$\leq \mu\{x \in \mathbb{R} : M(f-g) > \epsilon/3\} + \mu\{x \in \mathbb{R} : M(g-f) > \epsilon/3\}\,.$$

This in turn is

$$\leq 3 \cdot \frac{\|f-g\|_{L^1}}{\epsilon/3} + 3 \cdot \frac{\|g-f\|_{L^1}}{\epsilon/3} \leq 3\frac{\epsilon^2}{\epsilon/3} + 3\frac{\epsilon^2}{\epsilon/3} = 18\epsilon\,.$$

This proves that

$$\mu\left\{x \in \mathbb{R} : |\limsup_{R\to 0} \frac{1}{2R}\int_{-R}^{R} f(t)\,dt - \liminf_{R\to 0}\frac{1}{2R}\int_{-R}^{R} f(t)\,dt| > \epsilon\right\} = 0\,.$$

We have thus shown that the desired limit exists almost everywhere. Since, when f is continuous, it is obvious that the limit equals $f(x)$, it follows then that the limit equals f almost everywhere for any $f \in L^1$. $\square$

We conclude with a useful result about averages of functions against dilates of testing functions.

Let φ be a continuous function with compact support. Assume that $0 \leq \varphi \leq 1$ and that φ vanishes outside the interval $[-1, 1]$. Also suppose that $\int \varphi(x)\, dx = 1$. For $\epsilon > 0$ define $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(x/\epsilon)$. Then we have the following result. Recall that, if f, g are integrable functions, then

$$f * g(x) = \int f(x-t)g(t)\, dt\,.$$

Theorem: *Let f be a Lebesgue integrable function on $\mathbb{R}$. Then, for almost every $x \in \mathbb{R}$,*

$$\lim_{\epsilon \to 0} f * \varphi_\epsilon(x) = f(x)\,.$$

Proof: We observe that

$$\varphi(x) \le \frac{1}{1} \chi_{[-1,1]} \, .$$

As a result,

$$\varphi_\epsilon(x) \le \epsilon^{-1} \chi_{[-\epsilon,\epsilon]} \, .$$

Because of this last estimate, we see that the proof of this new theorem is just as in the proof of the last theorem. We estimate the limsup minus the liminf with three terms, and we then estimate each of those three terms with the corresponding term that comes from the Hardy-Littlewood maximal function. We leave the details for the interested reader. $\square$