Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

Metric Spaces

In your real analysis class, when you studied functions of several variables, you no doubt noticed many analogies with the one-variable theory. The arguments generally involved clever use of the triangle inequality. For functions of one variable, the inequality was for $|\ \ |$. For functions of several variables, the inequality was for $\|\ \ \|$.

This section formalizes a general context in which we may do analysis any time we have a reasonable notion of calculating distance. Such a structure will be called a metric:

Definition: A *metric space* is a pair (X, ρ), where X is a set and

$$\rho : X \times X \to \{t \in \mathbb{R} : t \geq 0\}$$

is a function satisfying

1. $\forall x, y \in X,\ \rho(x, y) = \rho(y, x)$;
2. $\rho(x, y) = 0$ if and only if $x = y$;
3. $\forall x, y, z \in X,\ \rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

The function ρ is called a *metric* on X.

Example: The pair $(\mathbb{R}, \rho)$, where $\rho(x, y) = |x - y|$, is a metric space. Each of the properties required of a metric is in this case a restatement of familiar facts from the analysis of one dimension.

The pair $(\mathbb{R}^k, \rho)$, where $\rho(x, y) = \|x - y\|$, is a metric space. Each of the properties required of a metric is in this case a restatement of familiar facts from the analysis of k dimensions.
□

The first example presented familiar metrics on two familiar spaces. Now we look at some new ones.

Example: The pair $(\mathbb{R}^2, \rho)$, where $\rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, is a metric space. Only the triangle inequality is not trivial to verify, but that reduces to the triangle inequality of one variable.

The pair $(\mathbb{R}, \mu)$, where $\mu(x, y) = 1$ if $x \neq y$ and 0 otherwise, is a metric space. Checking the triangle inequality reduces to seeing that if $x \neq y$ then either $x \neq z$ or $y \neq z$. □

Example: Let X denote the space of continuous functions on the interval $[0,1]$. If $f, g \in X$ then let $\rho(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$. Then the pair (X, ρ) is a metric space. The first two properties of a metric are obvious and the triangle inequality reduces to the triangle inequality for real numbers.

This example is a dramatic new departure from the analysis you have done in the past. For X is a very large space—infinite dimensional in a certain sense. Using the ideas that we are about to develop, it is nonetheless possible to study convergence, continuity, compactness, and the other basic concepts of analysis in this more general context. We shall see applications of these new techniques in later lectures. □

Now we begin to develop the tools of analysis in metric spaces.

Definition: Let (X, ρ) be a metric space. A sequence $\{x_j\}$ of elements of X is said to *converge* to a point $\alpha \in X$ if, for each $\epsilon > 0$, there is an $N > 0$ such that if $j > N$ then $\rho(x_j, \alpha) < \epsilon$. We call α the *limit* of the sequence $\{x_j\}$. We sometimes write $x_j \to \alpha$.

Compare this definition of convergence with the corresponding definition for convergence in the real line. Notice that it is identical, except that the sense in which distance is measured is now more general.

Example: Let (X, ρ) be the metric space from the last example, consisting of the continuous functions on the unit interval with the indicated metric function ρ. Then $f = \sin x$ is an element of this space, and so are the functions

$$f_j = \sum_{\ell=0}^{j} (-1)^\ell \frac{x^{2\ell+1}}{(2\ell+1)!} .$$

Observe that the functions f_j are the partial sums for the Taylor series of $\sin x$. We can check from simple estimates on the error term of Taylor's theorem that the functions f_j converge uniformly to f. Thus, in the language of metric spaces, $f_j \to f$ in the metric space notion of convergence. □

Definition: Let (X, ρ) be a metric space. A sequence $\{x_j\}$ of elements of X is said to be *Cauchy* if, for each $\epsilon > 0$ there is an $N > 0$ such that if $j, k > N$ then $\rho(x_j, x_k) < \epsilon$.

Now the Cauchy criterion and convergence are connected in the expected fashion:

Proposition: *Let $\{x_j\}$ be a convergent sequence, with limit α, in the metric space (X, ρ). Then the sequence $\{x_j\}$ is Cauchy.*

Proof: Let $\epsilon > 0$. Choose an N so large that, if $j > N$, then $\rho(x_j, \alpha) < \epsilon/2$. If $j, k > N$ then

$$\rho(x_j, x_k) \leq \rho(x_j, \alpha) + \rho(\alpha, x_k) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

That completes the proof. □

The converse of the proposition is true in the real numbers (with the usual metric). However, it is not true in every metric space. For example, the rationals $\mathbb{Q}$ with the usual metric $\rho(s, t) = |s - t|$ is a metric space; but the sequence

$$3\,,\,3.1\,,\,3.14\,,\,3.141\,,\,3.1415\,,\,3.14159,\ldots\,,$$

while certainly Cauchy, *does not converge to a rational number.* Instead it converges to π, which is irrational. Thus we are led to a definition:

Definition: We say that a metric space (X, ρ) is *complete* if every Cauchy sequence converges to an element of the metric space.

Thus the real numbers, with the usual metric, form a complete metric space. The rational numbers do not.

Example: Consider the metric space (X, ρ) from an earlier example, consisting of the continuous functions on the closed unit interval with the indicated metric function ρ. If $\{g_j\}$ is a Cauchy sequence in this metric space then each g_j is a continuous function on the unit interval and this sequence of continuous functions is Cauchy in the uniform sense. Therefore they converge uniformly to a limit function g that must be continuous. We conclude that the metric space (X, ρ) is complete. $\square$

Example: Consider the metric space (X, ρ) consisting of the polynomials, taken to have domain the interval $[0, 1]$, with the distance function $\rho(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$. This metric space is *not* complete. For if h is any continuous function on $[0, 1]$ that is not a polynomial, such as $h(x) = \sin x$ then, by the Weierstrass Approximation Theorem, there is a sequence $\{p_j\}$ of polynomials that converges uniformly on $[0, 1]$ to h. Thus this sequence $\{p_j\}$ will be Cauchy in the metric space, but it *does not converge to an element of the metric space.* We conclude that the metric space (X, ρ) is not complete. $\square$

If (X, ρ) is a metric space then an *(open) ball* with center $P \in X$ and radius r is the set

$$B(P, r) = \{x \in X : \rho(x, P) < r\} .$$

The *closed ball* with center P and radius r is the set

$$\overline{B}(P, r) = \{x \in X : \rho(x, P) \leq r\} .$$

Definition: Let (X, ρ) be a metric space and E a subset of X. A point $P \in E$ is called an *isolated point* of E if there is an $r > 0$ such that $E \cap B(P, r) = \{P\}$. If a point of E is not isolated then it is called *nonisolated.*

We see that the notion of “isolated” has intuitive appeal: an isolated point is one that is spaced apart—at least distance r—from the other points of the space. A nonisolated point, by contrast, has neighbors that are arbitrarily close.

Definition: Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$. If $P \in X$ is a nonisolated point and $\ell \in \mathbb{R}$ we say that *the limit of f at P is ℓ*, and write

$$\lim_{x \to P} f(x) = \ell \, ,$$

if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < \rho(x, P) < \delta$ then $|f(x) - \ell| < \epsilon$.

Notice in this definition that we use ρ to measure distance in X—that is the natural notion of distance with which X comes equipped—but we use absolute values to measure distance in $\mathbb{R}$.

The following lemma will prove useful.

Lemma: *Let (X, ρ) be a metric space and $P \in X$ a nonisolated point. Let f be a function from X to $\mathbb{R}$. Then $\lim_{x \to P} f(x) = \ell$ if and only if, for every sequence $\{x_j\} \subseteq X$ satisfying $x_j \to P$, it holds that $f(x_j) \to f(P)$.*

Proof: This is straightforward and is left for you to think about. $\square$

Definition: Let (X, ρ) be a metric space and E a subset of X. Suppose that $P \in E$. We say that a function $f : E \to \mathbb{R}$ is *continuous* at P if

$$\lim_{x \to P} f(x) = f(P) .$$

Example: Let (X, ρ) be the space of continuous functions on the interval $[0, 1]$ equipped with the supremum metric as in an earlier example. Define the function $\mathcal{F} : X \to \mathbb{R}$ by the formula

$$\mathcal{F}(f) = \int_0^1 f(t)\, dt\,.$$

Then $\mathcal{F}$ takes an element of X, namely a continuous function, to a real number, namely its integral over $[0, 1]$. We claim that $\mathcal{F}$ is continuous at every point of X.

For fix a point $f \in X$. If $\{f_j\}$ is a sequence of elements of X converging in the metric space sense to the limit f, then (in the language of classical analysis) the f_j are continuous functions converging uniformly to the continuous function f on the interval $[0,1]$. But, by a standard result from real analysis, it follows that

$$\int_0^1 f_j(t)\,dt \to \int_0^1 f(t)\,dt\,.$$

But this just says that $\mathcal{F}(f_j) \to \mathcal{F}(f)$. Using the lemma, we conclude that

$$\lim_{g \to f} \mathcal{F}(g) = \mathcal{F}(f)\,.$$

Therefore $\mathcal{F}$ is continuous at f.

Since $f \in X$ was chosen arbitrarily, we conclude that the function $\mathcal{F}$ is continuous at every point of X. $\square$

In the next lecture we shall develop some topological properties of metric spaces.