Math 4121
April 23, 2021 Lecture

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April 4, 2021

Figure: This is your instructor.

The Lebesgue Integral

Topology in a Metric Space

Fix a metric space (X, ρ). A set $U \subseteq X$ is called *open* if for each $u \in U$ there is an $r > 0$ such that $B(u, r) \subseteq U$. A set $E \subseteq X$ is called *closed* if its complement in X is open. Notice that these definitions are analogous to those that we gave in our discussion of topology in Math 4111.

Example: Consider the set of real numbers $\mathbb{R}$ equipped with the metric $\rho(s,t) = 1$ if $s \neq t$ and $\rho(s,t) = 0$ otherwise. Then each singleton $U = \{x\}$ is an open set. For let P be a point of U. Then $P = x$ and the ball $B(P, 1/2)$ lies in U.

However, each singleton is also closed. For the complement of the singleton $U = \{x\}$ is the set $S = \mathbb{R} \setminus \{x\}$. If $s \in S$ then $B(s, 1/2) \subseteq S$ as in the preceding paragraph. $\square$

Example: Let (X, ρ) be the metric space of continuous functions on the interval $[0, 1]$ equipped with the metric $\rho(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Define

$$U = \{f \in X : f(1/2) > 5\} .$$

Then U is an open set in the metric space. To verify this assertion, fix an element $f \in U$. Let $\epsilon = f(1/2) - 5 > 0$. We claim that the metric ball $B(f, \epsilon)$ lies in U. For let $g \in B(f, \epsilon)$.

Then

$$\begin{aligned} g(1/2) &= f(1/2) - |f(1/2) - g(1/2)| \\ &\geq f(1/2) - \rho(f, g) \\ &> f(1/2) - \epsilon \\ &= 5\,. \end{aligned}$$

It follows that $g \in U$. Since $g \in B(f, \epsilon)$ was chosen arbitrarily, we may conclude that $B(f, \epsilon) \subseteq U$. But this says that U is open.

We may also conclude from this calculation that

$$ {}^c U = \{ f \in X : f(1/2) \le 5 \} $$

is closed. $\square$

Topology in a Metric Space

Definition: Let (X, ρ) be a metric space and $S \subseteq X$. A point $x \in X$ is called an *accumulation point* of S (also called a limit point or a cluster point) if every $B(x, r)$ contains infinitely many elements of S.

Proposition: *Let (X, ρ) be a metric space. A set $S \subseteq X$ is closed if and only if every accumulation point of S lies in S.*

Proof: The proof is similar to the corresponding result in Math 4111 and we leave as an exercise for you. $\square$

Definition: Let (X, ρ) be a metric space. A subset $S \subseteq X$ is said to be *bounded* if S lies in some ball $B(P, r)$.

Definition: Let (X, ρ) be a metric space. A set $S \subseteq X$ is said to be *compact* if every sequence in S has a subsequence that converges *to an element of* S.

Example: In Math 4111 we learned that, in the real number system, compact sets are closed and bounded, and conversely. Such is not the case in general metric spaces.

As an example, consider the metric space (X, ρ) consisting of all continuous functions on the interval $[0, 1]$ with the supremum metric as in previous examples. Let

$$S = \{f_j(x) = x^j : j = 1, 2, \dots\} .$$

This set is bounded since it lies in the ball $B(\mathbf{0}, 2)$ (here $\mathbf{0}$ denotes the identically zero function). We claim that S contains no Cauchy sequences.

This follows (see the discussion of uniform convergence in Math 4111) because, no matter how large N is, if $k > j > N$ then we may write

$$|f_j(x) - f_k(x)| = \left|x^j\right| \left|(x^{k-j} - 1)\right| .$$

Fix j. If x is sufficiently near to 1 then $|x^j| > 3/4$. But then we may pick k so large that $|x^{k-j}| < 1/4$. Thus

$$|f_k(x) - f_j(x)| \geq 9/16 .$$

So there is no Cauchy subsequence. We may conclude (for vacuous reasons) that S is closed.

But S is not compact. For, as just noted, the sequence $\{f_j\}$ consists of infinitely many distinct elements of S which do not have a convergent subsequence (indeed not even a Cauchy subsequence). $\square$

In spite of the last example, half of the Heine-Borel theorem is true:

Proposition: *Let (X, ρ) be a metric space and S a subset of X. If S is compact then S is closed and bounded.*

Proof: Let $\{s_j\}$ be a Cauchy sequence in S. By compactness, this sequence must contain a subsequence converging to some limit P. But since the full sequence is Cauchy, the full sequence must converge to P (exercise). Thus S is closed.

If S is not bounded, we derive a contradiction as follows. Fix a point $P_1 \in S$. Since S is not bounded we may find a point P_2 that has distance at least 1 from P_1. Since S is unbounded, we may find a point P_3 of S that is distance at least 2 from both P_1 and P_2. Continuing in this fashion, we select $P_j \in S$ which is distance at least j from $P_1, P_2, \ldots P_{j-1}$. Such a sequence $\{P_j\}$ can have no Cauchy subsequence, contradicting compactness. Therefore S is bounded. $\square$

Definition: Let S be a subset of a metric space (X, ρ). A collection of open sets $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ (each $\mathcal{O}_\alpha$ is an open set in X) is called an *open covering* of S if

$$\cup_{\alpha \in A} \mathcal{O}_\alpha \supseteq S\,.$$

Definition: If $\mathcal{C}$ is an open covering of a set S and if $\mathcal{D}$ is another open covering of S such that each element of $\mathcal{D}$ is also an element of $\mathcal{C}$ then we call $\mathcal{D}$ a *subcovering* of $\mathcal{C}$.

We call $\mathcal{D}$ a *finite subcovering* if $\mathcal{D}$ has just finitely many elements.

Theorem: *A subset S of a metric space (X, ρ) is compact if and only if every open covering $\mathcal{C} = \{\mathcal{O}_\alpha\}_{\alpha \in A}$ of S has a finite subcovering.*

Proof: The forward direction is beyond the scope of this treatment and we shall not discuss it.

The proof of the converse direction is similar in spirit to the proof in Math 4111. We leave the details for you to think about. $\square$

Proposition: *Let S be a compact subset of a metric space (X, ρ). If E is a closed subset of S then E is compact.*

Proof: Let $\mathcal{C}$ be an open covering of E. The set $U = X \setminus E$ is open and the covering $\mathcal{C}'$ consisting of all the open sets in $\mathcal{C}$ together with the open set U covers S. Since S is compact we may find a finite subcovering

$$O_1, O_2, \ldots O_k$$

that covers S. If one of these sets is U then discard it. The remaining $k - 1$ open sets cover E. □

Definition: If (X, ρ) is a metric space and $E \subseteq X$ then the *closure* of E is defined to be the union of E with the set of its accumulation points.