Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

The Baire Category Theorem

This lecture is a dramatic conclusion to our efforts this semester, and this year. For it demonstrates how abstract thinking can clarify matters and can yield results that one cannot even imagine in the more classical, concrete context. The subject of our discussion today is the *Baire category theorem*. There is actually quite a lot of theory based on Baire's ideas; this will only be an introduction.

Let (X, ρ) be a metric space and $S \subseteq X$ a subset. A set $E \subseteq X$ is said to be *dense* in S if every element of S is the limit of some sequence of elements of E.

Example: The set of rational numbers $\mathbb{Q}$ is dense in any nontrivial interval of $\mathbb{R}$. $\square$

Example: Let (X, ρ) be the metric space of continuous functions on the interval $[0, 1]$ equipped with the supremum metric as usual. Let $E \subseteq X$ be the polynomial functions. Then the Weierstrass Approximation Theorem tells us that E is dense in X. $\square$

Example: Consider the real numbers $\mathbb{R}$ with the metric $\rho(s, t) = 1$ if $s \neq t$ and $\rho(s, t) = 0$ otherwise. Then no proper subset of $\mathbb{R}$ is dense in $\mathbb{R}$. To see this, notice that if E were dense and were not all of $\mathbb{R}$ and if $P \in \mathbb{R} \setminus E$ then $\rho(P, e) > 1/2$ for all $e \in E$. So elements of E do not get close to P. Thus E is not dense in $\mathbb{R}$. $\square$

Refer to an earlier definition for the concept of closure of a set.

Example: Let (X, ρ) be the set of real numbers with the usual metric and set $E = \mathbb{Q} \cap (-2, 2)$. Then the closure of E is $[-2, 2]$.

Let (Y, σ) be the continuous functions on $[0, 1]$ equipped with the supremum metric as in our earlier examples. Take $E \subseteq Y$ to be the polynomials. Then the closure of E is Y. $\square$

We note in passing that, if $B(P, r)$ is a ball in a metric space (X, ρ), then $\overline{B}(P, r)$ will contain but need not be equal to the closure of $B(P, r)$ (for which see an exercise in the last section).

Definition: Let (X, ρ) be a metric space. We say that $E \subseteq X$ is *nowhere dense* in X if the closure of E contains no ball $B(x, r)$ for any $x \in X, r > 0$.

Example: Let us consider the integers $\mathbb{Z}$ as a subset of the metric space $\mathbb{R}$ equipped with the standard metric. Then the closure of $\mathbb{Z}$ is $\mathbb{Z}$ itself. And of course $\mathbb{Z}$ contains no metric balls. Therefore $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$. $\square$

Example: Consider the metric space X of all continuous functions on the unit interval $[0, 1]$, equipped with the usual supremum metric. Fix $k > 0$ and consider

$$E \equiv \{p(x) : p \text{ is a polynomial of degree not exceeding } k\} .$$

Then the closure of E is E itself (that is, the limit of a sequence of polynomials of degree not exceeeding k is still a polynomial of degree not exceeding k—details are requested of you in the exercises). And E contains no metric balls. For if $p \in E$ and $r > 0$ then $p(x) + (r/2) \cdot x^{k+1} \in B(p, r)$ but $p(x) + (r/2) \cdot x^{k+1} \notin E$.

We recall, as noted in an earlier example, that the set of *all* polynomials is dense in X; but if we restrict attention to polynomials of degree not exceeding a fixed number k then the resulting set is nowhere dense. $\square$

Theorem (The Baire Category Theorem): *Let (X, ρ) be a complete metric space. Then X cannot be written as the union of countably many nowhere dense sets.*

Proof: This proof is quite similar to the proof that we presented in Math 4111 that a perfect set must be uncountable. You may wish to review that proof at this time.

Seeking a contradiction, suppose that X may be written as a countable union of nowhere dense sets $Y_1, Y_2, \ldots$. Choose a point $x_1 \in {}^c\overline{Y}_1$. Since Y_1 is nowhere dense we may select an $r_1 > 0$ such that $\overline{B}_1 \equiv \overline{B}(x_1, r_1)$ satisfies $\overline{B}_1 \cap \overline{Y}_1 = \emptyset$. Assume without loss of generality that $r_1 < 1$.

Next, since Y_2 is nowhere dense, we may choose $x_2 \in \overline{B}_1 \cap {}^c\overline{Y}_2$ and an $r_2 > 0$ such that $\overline{B}_2 = \overline{B}(x_2, r_2) \subseteq \overline{B}_1 \cap {}^c\overline{Y}_2$. Shrinking B_2 if necessary, we may assume that $r_2 < \frac{1}{2} r_1$. Continuing in this fashion, we select at the jth step a point $x_j \in \overline{B}_{j-1} \cap {}^c\overline{Y}_j$ and a number $r_j > 0$ such that $r_j < \frac{1}{2} r_{j-1}$ and $\overline{B}_j = \overline{B}(x_j, r_j) \subseteq \overline{B}_{j-1} \cap {}^c\overline{Y}_j$.

Now the sequence $\{x_j\}$ is Cauchy since all the terms x_j for $j > N$ are contained in a ball of radius $r_N < 2^{-N}$ hence are not more than distance 2^{-N} apart. Since (X, ρ) is a complete metric space, we conclude that the sequence converges to a limit point P. Moreover, by construction, $P \in \overline{B}_j$ for every j hence is in the complement of *every* $\overline{Y}_j$. Thus $\bigcup_j Y_j \neq X$. That is a contradiction, and the proof is complete. $\square$

There is quite a lot of terminology associated with the Baire theorem, and we shall not detail it all here. We do recall that a G_δ is the countable intersection of open sets.

Before we apply the Baire Category Theorem, let us formulate some restatements, or corollaries, of the theorem which follow immediately from the definitions.

Corollary: *Let (X, ρ) be a complete metric space. Let $Y_1, Y_2, \ldots$ be countably many closed subsets of X, each of which contains no nontrivial open ball. Then $\bigcup_j Y_j$ also has the property that it contains no nontrivial open ball.*

Corollary: *Let (X, ρ) be a complete metric space. Let $O_1, O_2, \ldots$ be countably many dense open subsets of X. Then $\bigcap_j O_j$ is dense in X.*

Note that the result of the second corollary follows from the first corollary by complementation. The set $\bigcap_j O_j$, while dense, need not be open.

Example: The metric space $\mathbb{R}$, equipped with the standard Euclidean metric, cannot be written as a countable union of nowhere dense sets. $\square$

By contrast, $\mathbb{Q}$ *can* be written as the union of the singletons $\{q_j\}$ where the q_j represent an enumeration of the rationals. Each singleton is of course nowhere dense since it is the limit of other rationals in the set. However, $\mathbb{Q}$ is not complete.

Example: Baire's theorem contains the fact that a perfect set of real numbers must be uncountable. For if P were perfect and countable we could write $P = \{p_1, p_2, \dots\}$. Therefore

$$P = \bigcup_{j=1}^{\infty} \{p_j\}\,.$$

But each of the singletons $\{p_j\}$ is a nowhere dense set in the metric space P. And P is complete. (You should verify both these assertions for yourself.) This contradicts the Category Theorem. So P cannot be countable. $\square$

A set that can be written as a countable union of nowhere dense sets is said to be of *first category*. If a set is not of first category, then it is said to be of *second category*. The Baire Category Theorem says that a complete metric space must be of second category. We should think of a set of first category as being "thin" and a set of second category as being "fat" or "robust." (This is one of many ways that we have in mathematics of distinguishing "fat" sets. Countability and uncountability is another. Lebesgue's measure theory is a third.)

One of the most striking applications of the Baire Category Theorem is the following result to the effect that "most" continuous functions are nowhere differentiable. This explodes the myth that most of us mistakenly derive from calculus class that a typical continuous function is differentiable at all points except perhaps at a discrete set of bad points.

Theorem: *Let (X, ρ) be the metric space of continuous functions on the unit interval $[0, 1]$ equipped with the metric*

$$\rho(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)| .$$

Define a subset of E of X as follows: $f \in E$ if there exists one point at which f is differentiable. Then E is of first category in the complete metric space (X, ρ).

Proof: For each pair of positive integers m, n we let

$$A_{m,n} = \{f \in X : \exists x \in [0,1] \text{ such that } |f(x) - f(t)| \leq n|x - t|$$

$$\forall\ t \in [0,1] \text{ that satisfy } |x - t| \leq 1/m\}\,.$$

Fix m and n. We claim that $A_{m,n}$ is nowhere dense in X. In fact, if $f \in A_{m,n}$ set

$$K_f = \max_{x \in [0,1]} \left| \frac{f(x \pm 1/m) - f(x)}{1/m} \right|\,.$$

Let $h(x)$ be a continuous piecewise linear function, bounded by 1, consisting of linear pieces having slope $3K_f$. Then for every $\epsilon > 0$ it holds that $f + \epsilon \cdot h$ has metric distance less than ϵ from f and is not a member of $A_{m,n}$. This proves that $A_{m,n}$ is nowhere dense.

We conclude from Baire's theorem that $\cup_{m,n} A_{m,n}$ is nowhere dense in X. Therefore $S = X \setminus \cup_{m,n} A_{m,n}$ is of second category. But if $f \in S$ then for every $x \in [0,1]$ and every $n > 0$ there are points t arbitrarily close to x (that is, at distance $\leq 1/m$ from x) such that

$$\left| \frac{f(x) - f(t)}{t - x} \right| > n \, .$$

It follows that f is differentiable at no $x \in [0,1]$. That proves the assertion. $\square$

Again, this is an exciting way to end the course—with a theorem that we could not have imagined a couple of weeks ago. This is the power of mathematics.