Math 4121 Spring, 2018
Krantz

Final Exam

General Instructions: Write out the solution to each problem on standard 8.5 in. × 11 in. paper. Make your solutions as complete as possible. Leave nothing to the imagination.

You may consult your notes or the textbook or any book in the library. Of course you may talk to me about the exam. But you should *not* talk to each other or to anyone else.

This exam is worth 115 points. It is due on Monday, May 7, 2018.

(8 points) **1.** Let f_j be continuous functions on the interval $[0, 1]$ and suppose that $f_j \to f$ uniformly on the interval. Then, using Lebesgue measure μ and the Lebesgue integral, prove that

$$\lim_{j\to+\infty} \int f_j d\mu = \int \lim_{j\to+\infty} f_j d\mu \,.$$

(8 points) **2.** Are the simple functions dense in $L^\infty(\mathbb{R})$ (in the L^∞ topology)? Why or why not?

(10 points) **3.** Let X be a Banach space. Prove that X can be realized as a subset of $(X^*)^*$ in a natural way.

(9 points) **4.** Let X and Y be Banach spaces. Show that $X \times Y$ is a Banach space when equipped with a suitable norm. What should that norm be? Prove your assertions.

(10 points) **5.** The greatest integer function g is defined as follows:

$$g(x) = \text{the greatest integer less than or equal to } x \,.$$

Observe that g is monotone increasing and right continuous. So, by the Lebesgue-Stieltjes theory, it induces a measure γ. Explicitly calculate

$$\int_{[-11/2,13/2]} x^2 \, d\gamma(x) \, .$$

[**Hint:** Your answer should be a *number*.]

(10 points) **6.** Let f be an L^p function on the real line with $1 \leq p < \infty$. Let μ be Lebesgue measure. Define, for $t \geq 0$,

$$\lambda_f(t) = \mu\Big(\{x \in \mathbb{R} : |f(x)| > t\}\Big) \, .$$

Prove that

$$\int |f(x)|^p \, d\mu = \int_0^\infty p \cdot t^{p-1} \cdot \lambda_f(t) \, dt \, .$$

[**Hint:** Use Fubini's theorem (Chapter 13).]

(10 points) **7.** If g is a monotone increasing, right continuous function then, by the Lebesgue-Stieltjes theory, it induces a measure γ. Rather than write

$$\int f \, d\gamma$$

it is often convenient to write

$$\int f \, dg \, .$$

It is straightforward to check that $\int f \, dg$ is the limit of the Riemann-Stieltjes sums

$$\sum_{j=1}^{k} f(\xi_j) \cdot \Big(g(x_j) - g(x_{j-1})\Big)$$

for ξ_j a point in the interval (x_{j-1}, x_j). [**Note:** You need not prove this last assertion. Just accept it as given.]

If f, g are both monotone increasing functions that are continuous, then prove that

$$\int_a^b f \, dg = \Big[f(b)g(b) - f(a)g(a)\Big] - \int_a^b g \, df \, .$$

This is a version of integration by parts in the Stieltjes context.

(10 points) **8.** Let μ be Lebesgue measure. Construct a sequence of continuous functions f_j on $[0,1]$ such that $0 \le f_j(x) \le 1$ for all $x \in [0,1]$ and all j and so that

$$\lim_{j\to+\infty} \int_0^1 f_j(x)\, d\mu = 0\,,$$

but so that the sequence $\{f_j(x)\}$ converges for no $x \in [0,1]$.

(10 points) **9.** Let $\{f_j\}$ be a sequence of continuous functions on $[0,1]$ such that $0 \le f_j(x) \le 1$, for all $x \in [0,1]$ and all j, such that $f_j(x) \to 0$ as $j \to +\infty$ for every $x \in [0,1]$. Prove that

$$\lim_{j\to+\infty} \int_0^1 f_j(x)\, d\mu = 0\,,$$

where μ is Lebesgue measure.

(10 points) **10.** Let μ be Lebesgue measure. Construct a Borel set $E \subseteq [0,1]$ so that

$$0 < \mu(E \cap I) < \mu(I)$$

for every interval $I \subseteq [0,1]$ of positive length.

(10 points) **11.** Prove that a measure μ on the real line is σ-finite if and only if there is an $f \in L^1(\mathbb{R}, \mu)$ such that $f(x) > 0$ for every $x \in \mathbb{R}$.

(10 points) **12.** The *support* of a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be the closure of the set of points at which f does not vanish. Is it true that any compact subset $K \subseteq \mathbb{R}$ is the support of some function that is continuous on $\mathbb{R}$? Why or why not?