DATE MATH 4121
PURPOSE Spring, 2021 1

SOLUTIONS TO HW 7

CUE COLUMN NOTES

Chapter 11

1. Simply take the closures of the open intervals I_j found in the proof in the text.

5. Since $\mathcal{O}$ is open, $\mathcal{O}$ is the countable disjoint union of open intervals I_j. Write

$$I_j = (a_j, b_j),$$

where a_j could be $-\infty$ or b_j could be $+\infty$.

First assume a_j, b_j are both finite for every j. Let $\varepsilon > 0$. Set

$$J_j^{\varepsilon} = \left[a_j + \frac{\varepsilon}{2^j},\ b_j - \frac{\varepsilon}{2^j}\right].$$

For K a positive integer, let

$$E_k^{\varepsilon} = \bigcup_{j=1}^{K} J_j^{\varepsilon}.$$

Then E_K^{ε} is compact and

$$\bigcup_{l=1}^{\infty} \bigcup_{K=1}^{\infty} E_K^{2^{-l}} = \mathcal{O}.$$

If some a_j equals $-\infty$, replace J_j^{ε} by

$$J_j = \left[-j,\ b_j - \frac{\varepsilon}{2^j}\right]$$

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Similarly if some $b_j = +\infty$.

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Chapter 12

1. $f_j(x) = j^{-1/p}\chi_{[0,j]}(x)$.

$|f_j(x) - 0| \le j^{-1/p} \to 0$ as $j \to \infty$.

So $f_j \to 0$ uniformly.

$\left(\int |f_j - 0|^p dx\right)^{1/p} = \int_0^j (j^{-1/p})^p dx = 1 \not\to 0$.

So $f_j \not\to 0$ in L^p.

4. $h_j(x) = \chi_{[j,j+1]}$. Let $x \in \mathbb{R}$.

If $j > x$ then $h_j(x) = 0$. So $h_j \to 0$ pointwise.

Let $\alpha = 1/2$.

$\mu\{x : |h_j(x) - 0| \ge \alpha\} = 1$

for every j. Hence $\mu\{x : |h_j(x) - 0| \ge \alpha\} \not\to 0$.

So does not converge in measure.

Chapter 13.

1. If both A and B are non-empty then let $a \in A$, $b \in B$. Then $(a,b) \in A \times B$ so $A \times B \neq \emptyset$. If either A or B is empty then $A \times B$ is empty.

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3. Let $A = [a_1, a_2]$, $B = [b_1, b_2]$.

Then $\mathbb{R} \times \mathbb{R} \setminus (A \times B) =$

$[a_1, a_2] \times (-\infty, b_1] \cup [a_1, a_2] \times [b_2, \infty)$

$\cup [a_2, \infty] \times [b_1, b_2] \cup [-\infty, a_1] \times [b_1, b_2]$.

Chapter 14

1. Let $S_0 = [0, 1]$

$S_1 = [0, 1] \setminus \left(\frac{1}{2} - \frac{1}{5}, \frac{1}{2} + \frac{1}{5}\right)$

$S_2 = S_1 \setminus \left[\left(\frac{1}{4} - \frac{1}{25}, \frac{1}{4} + \frac{1}{25}\right) \cup \left(\frac{3}{4} - \frac{1}{25}, \frac{3}{4} + \frac{1}{25}\right)\right]$

etc.

Then, as we know, $S = \bigcap S_j$ is a Cantor-like set of positive measure. But S has no subintervals.

3. Let S be a null set. Define

$S_j = S \cap [j, j+1]$.

Then each S_j is a null set and

$\bigcup_j S_j = S$.

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Chapter 15

3. $\mathbb{Q} \ominus \mathbb{Q} = \{q_1 - q_2 : q_1 \in \mathbb{Q}, q_2 \in \mathbb{Q}\}$.

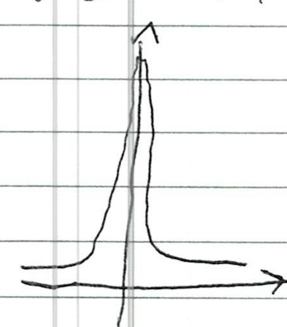
If $q \in \mathbb{Q}$ then $q = (q+q) - q$. So
$q \in \mathbb{Q} \ominus \mathbb{Q}$.

If $r \in \mathbb{Q} \ominus \mathbb{Q}$, then $r = s - t$
for $s \in \mathbb{Q}$, $t \in \mathbb{Q}$ so $r \in \mathbb{Q}$.
Thus $\mathbb{Q} \ominus \mathbb{Q} = \mathbb{Q}$.

7. Any subset of a null set is also a measurable null set.

Chapter 16

3. The graph of φ_ε looks like this:



As $\varepsilon \to 0$, φ_ε approximates the Dirac delta function.

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5. Let $f(x) = x^2$. If K is a compact set then K is a subset of a bounded interval $[a,b]$. So

$$\int_K x^2 dx \le \int_a^b x^2 dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

But f is <u>not</u> integrable.

BONUS PROBEM A:

If $(x_1,y_1), (x_2,y_2) \in X \times Y$, define

$$d((x_1,y_1),(x_2,y_2)) = \max\{\rho(x_1,x_2), \sigma(y_1,y_2)\}.$$

Then d is a metric on $X \times Y$.

BONUS PROBLEM B:

This is <u>not</u> a metric. Let
$f(x) = x$, $g(x) = x+2$. Then $f \neq g$
but
$\rho(f,g) = 0$.

BONUS PROBLEM C:

Let $X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\} \cup \{(0,0)\}$.
Then $B((0,0),1) = \{(0,0)\}$.

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$$\overline{B((0,0),1)} = X.$$

And $\overline{B}((0,0),1)$ is <u>not</u> the closure of $B((0,0),1)$.

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BONUS PROBLEM D: Any union of intervals

$I = \{x \in \mathbb{Q} : a < x < b\}$.

Here a, b could be finite or $\pm\infty$.

BONUS PROBLEM E:

The union of the E_j is dense in X but it does not equal X.

Similarly, the union of the singleton rationals is dense in $\mathbb{R}$ but does not equal $\mathbb{R}$.

BONUS PROBLEM F:

There are many solutions to this problem. Here is one:

If $k = 0$ then each p_j is constant. Obviously the pointwise limit of constant functions is constant.

If $k = 1$ then each p_j is linear. But then each $q_j(x) \equiv p_j(x+1) - p_j(x)$ is constant. So the previous step applies and the limit of the q_j is constant. So the limit of the p_j is linear.

And now we can handle the general case by induction on k.

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