Math 4121 Spring, 2018

SOLUTIONS TO HW 1

1. Let $x \in [a,b]$. Then, for $j = 1, 2, \ldots$,

$a - \frac{1}{j} < x < b + 1/j$. So

$$x \in \bigcap_{j=1}^{\infty} (a - 1/j, b + 1/j).$$

Conversely, let $x \in \bigcap_{j=1}^{\infty} (a - \frac{1}{j}, b + \frac{1}{j})$. So

$$a - \frac{1}{j} < x < b + \frac{1}{j} \quad \forall j = 1, 2, \ldots.$$

It follows that $a \le x \le b$. So $x \in [a,b]$.

Now let $x \in (a,b)$. Then $a < x < b$.

So $\exists \ell \in \mathbb{N}$ such that $a < x - \frac{1}{\ell} < x < x + \frac{1}{\ell} < b$.

Then $x \in [a + \frac{1}{\ell}, b - \frac{1}{\ell}]$. Therefore

$$x \in \bigcup_{j=1}^{\infty} [a + \frac{1}{j}, b - \frac{1}{j}].$$

Conversely, if $x \in \bigcup_{j=1}^{\infty} [a + \frac{1}{j}, b - \frac{1}{j}]$

Then $a + \frac{1}{j} \le x \le b - \frac{1}{j}$ for some $j = 1, 2, \ldots$.

Thus $x \in (a, b)$.

3. If x belongs to infinitely many of the B_n, then, for each k, there is an $\ell > k$ such that $x \in B_\ell$. So $x \in \bigcup_{j=k}^{\infty} B_j$. Since this is true for every k,

$$x \in \bigcap_{k=1}^{\infty} \left[\bigcup_{j=k}^{\infty} B_j \right].$$

Conversely, if $x \in \bigcap_{k=1}^{\infty} \left[\bigcup_{j=k}^{\infty} B_j \right]$ then, for every $k \geq 1$, $\exists\, j \geq k$ such that $x \in B_j$. But this just says that x lies in infinitely many of the B_j.

5. Let S be the non-measurable set constructed in the proof of Theorem 1.7. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in S \\ -1 & \text{if } x \notin S. \end{cases}$$

Then f is not Borel measurable. But $f^2 \equiv 1$ and $|f| \equiv 1$ <u>are</u> both Borel measurable.

6. Let f be Borel measurable and set

$$F_M(x) = \begin{cases} f(x) & \text{if } |f(x)| \le M \\ M & \text{if } f(x) > M \\ -M & \text{if } f(x) < -M. \end{cases}$$

a) If $\alpha > M$, then

$$F_M^{-1}((\alpha, \infty)) = \phi.$$

b) If $\alpha = M$, then

$$F_M^{-1}((\alpha, \infty)) = \phi.$$

c) If $-M < \alpha < M$, then

$$F_M^{-1}((\alpha, \infty)) = \{x : \alpha < f(x) \le M\} \cup \{x : f(x) > M\}$$

d) If $\alpha = -M$, then

$$F_M^{-1}((\alpha, \infty)) = \{x : f(x) > M\} \cup \{x : |f(x)| \le M\} \setminus \{x : f(x) = -M\}$$

e) If $\alpha < -M$, then

$$F_M^{-1}((\alpha, \infty)) = \mathbb{R}.$$

All these sets are Borel because f is Borel.

8. Let $x \in f^{-1}(\phi)$. So $f(x) \in \phi$. No such x exists. Thus $f^{-1}(\phi) \subseteq \phi$.
Let $x \in \phi$. Then $x \in f^{-1}(\phi)$ because a statement of the form $F \to F$ is always true.
Thus $f^{-1}(\phi) = \phi$.

Now let $x \in f^{-1}(\mathbb{R})$. So $f(x) \in \mathbb{R}$. Hence $x \in \mathbb{R}$. If instead $x \in \mathbb{R}$ then $f(x) \in \mathbb{R}$ so $x \in f^{-1}(\mathbb{R})$. We conclude that $f^{-1}(\mathbb{R}) = \mathbb{R}$.

If $x \in f^{-1}(S \setminus T)$ then $f(x) \in S \setminus T$ so $f(x) \in S$, $f(x) \notin T$. Thus $x \in f^{-1}(S)$ and $x \notin f^{-1}(T)$. In conclusion, $x \in f^{-1}(S) \setminus f^{-1}(T)$.
Conversely, if $x \in f^{-1}(S) \setminus f^{-1}(T)$ then $f(x) \in S$, $f(x) \notin T$, So $f(x) \in S \setminus T$ hence $x \in f^{-1}(S \setminus T)$. Thus $f^{-1}(S \setminus T) = f^{-1}(S) \setminus f^{-1}(T)$.

If $x \in f^{-1}(\bigcup_{j=1}^{\infty} F_j)$ then $f(x) \in \bigcup_{j=1}^{\infty} F_j$ so $f(x) \in F_j$ for some j. Hence $x \in f^{-1}(F_j)$. Therefore $x \in \bigcup_{j=1}^{\infty} f^{-1}(F_j)$. Conversely, if $x \in \bigcup_{j=1}^{\infty} f^{-1}(F_j)$ then $x \in f^{-1}(F_j)$ for some j hence $f(x) \in F_j$ for some j. Thus $f(x) \in \bigcup_{j=1}^{\infty} F_j$ so that $x \in f^{-1}(\bigcup_{j=1}^{\infty} F_j)$.

We conclude that

$$f^{-1}\left(\bigcup_{j=1}^{\infty} F_j\right) = \bigcup_{j=1}^{\infty} f^{-1}(F_j).$$

If we accept that

$$f^{-1}({}^cS) = {}^c(f^{-1}(S))$$

(which is easily verified), then the second inequality follows from de Morgan's Law.

9. If U is open then $f^{-1}(U)$ is Borel. If E is closed then let $U = {}^cE$ open.

Then $f^{-1}(E) = f^{-1}({}^cU) = {}^c(f^{-1}(U))$.

Since $f^{-1}(U)$ is Borel, its complement is also Borel. By Exercise 8, the inverse operation respects countable unions and intersections. The result follows.

11. $\mathcal{X}$ consists of those sets which are either denumerable or have denumerable complement.

a) ϕ is denumerable
$^c\mathbb{R} = \phi$ is denumerable

b) If $A \in \mathcal{X}$ then
(i) If A is denumerable, then ^cA has denumerable complement.
(ii) If A has denumerable complement, then ^cA is denumerable.

c) If $A_j \in \mathcal{X}$ then there are two possibilities
i) Each A_j is denumerable. Then $\bigcup_j A_j$ is denumerable.
ii) Some A_{j_0} has denumerable complement.
Then
$$^c\left(\bigcup_j A_j\right) = \bigcap_j {}^cA_j \subseteq {}^cA_{j_0}.$$
Since $^cA_{j_0}$ is denumerable, we conclude that $^c\left(\bigcup_j A_j\right)$ is denumerable.