HOMEWORK 2 Solutions

1. Define $\lambda(E) = \mu(K \cap E)$.

Then a) $\lambda(\phi) = \mu(K \cap \phi) = \mu(\phi) = 0$.

b) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{X}$, then

$$\lambda\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(K \cap \left(\bigcup_{j=1}^{\infty} E_j\right)\right)$$

$$= \mu\left(\bigcup_{j=1}^{\infty} (K \cap E_j)\right)$$

$$= \sum_{j=1}^{\infty} \mu(K \cap E_j)$$

$$= \sum_{j=1}^{\infty} \lambda(E_j).$$

So λ is a measure.

2. Set $$\mu = \sum_{j=1}^{k} a_j \mu_j$$

Then

a) $$\mu(\phi) = \sum_{j=1}^{k} a_j \mu_j(\phi) = \sum_{j=1}^{k} a_j \cdot 0 = 0.$$

b) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{X}$, then

$$\mu\left(\bigcup_{\ell=1}^{\infty} E_\ell\right) = \sum_{j=1}^{k} a_j \mu_j\left(\bigcup_{\ell=1}^{\infty} E_\ell\right)$$
$$= \sum_{j=1}^{k} a_j \left(\sum_{\ell=1}^{\infty} \mu_j(E_\ell)\right)$$
$$= \sum_{\ell=1}^{\infty} \sum_{j=1}^{k} a_j \mu_j(E_\ell)$$
$$= \sum_{\ell=1}^{\infty} \mu(E_\ell).$$

4. $$\mu(\liminf E_j) = \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j\right)$$
$$= \sup_k \mu\left(\bigcap_{j=k}^{\infty} E_j\right)$$
$$\le \sup_k \inf_j \mu(E_j)$$
$$= \liminf_{j\to\infty} \mu(E_j).$$

5. $\mu(\{p\}) \le \mu((p-\varepsilon, p+\varepsilon)) = 2\varepsilon \quad \forall \varepsilon > 0.$

Hence $\mu(\{p\}) = 0$.

If $a_1, a_2, \ldots$ is a countable set, then

$$\mu(\{a_j\}) \le \mu\left(\bigcup_{j=1}^{\infty} (a_j - 2^{-j}\varepsilon, a_j + 2^{-j}\varepsilon)\right) \quad \forall \varepsilon > 0$$

$$\le \sum_{j=1}^{\infty} \mu((a_j - 2^{-j}\varepsilon, a_j + 2^{-j}\varepsilon))$$

$$= \sum_{j=1}^{\infty} 2^{-j+1}\varepsilon$$

$$= 2\varepsilon.$$

Hence $\mu(\{a_j\}) = 0$.

6. For each $k \in K$, let $(k - 1/2, k + 1/2) = I_k$ be an open interval centered at k. Then $\bigcup_k I_k \supseteq K$. So there is a finite subcover $I_{k_1}, I_{k_2}, \ldots, I_{k_p}$. Then

$$\mu(K) \le \mu\left(\bigcup_{j=1}^{p} I_{k_j}\right) \le \sum_{j=1}^{p} \mu(I_{k_j}) = \sum_{j=1}^{p} 1 = p.$$

8. Define a new Cantor set as follows:

$$C_0 = [0, 1]$$

$$C_1 = C_0 \setminus \left[\frac{1}{2} - \frac{1}{10}, \frac{1}{2} + \frac{1}{10}\right]$$

$$C_2 = C_1 \setminus \left(\left[\frac{2}{10} - \frac{1}{50}, \frac{2}{10} + \frac{1}{50}\right] \cup \left[\frac{8}{10} - \frac{1}{50}, \frac{8}{10} + \frac{1}{50}\right]\right)$$

etc.

C_j is defined by set-theoretically subtracting from C_{j-1} a total of 2^{j-1} closed intervals each of length $\frac{1}{5^j}$. Set $C = \bigcap_{j=1}^{\infty} C_j$.

Then the measure of the union of all the removed intervals is

$$\sum_{j=1}^{\infty} 2^{j-1} \cdot \frac{1}{5^j} = \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{2}{5}\right)^j = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{2}{5}\right)^{j+1}$$

$$= \frac{1}{5} \sum_{j=0}^{\infty} \left(\frac{2}{5}\right)^j = \frac{1}{5} \cdot \frac{1}{1 - 2/5} = \frac{1}{5} \cdot \frac{1}{3/5}$$

$$= \frac{1}{5} \cdot \frac{5}{3} = \frac{1}{3}.$$

Thus the length of the new Cantor set is $2/3$.

It is clear that C_j contains no interval that is any longer than 2^{-j+1}. Thus the Cantor set, defined as $\bigcap_{j=1}^{\infty} C_j$ contains no intervals.

9. By Exercise 5, the measure of $\mathbb{Q}$ is 0. Hence the measure of the set of irrationals is $+\infty$.

11. Set

$$\lambda(E) = \mu(f^{-1}(E)).$$

a) $\lambda(\phi) = \mu(f^{-1}(\phi)) = \mu(\phi) = 0$.

b) If E_j are pairwise disjoint sets in X, then

$$\lambda\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(f^{-1}\left(\bigcup_{j=1}^{\infty} E_j\right)\right) = \mu\left(\bigcup_{j=1}^{\infty} f^{-1}(E_j)\right)$$

Now each $f^{-1}(E_j)$ is Borel. And the $f^{-1}(E_j)$ are pairwise disjoint.

So this last equals

$$\sum_{j=1}^{\infty} \mu(f^{-1}(E_j))$$

$$= \sum_{j=1}^{\infty} \lambda(E_j).$$