# SOLUTIONS TO HW3

1. Let $f = \sum_{j=1}^{k} a_j \chi_{E_j}$ and $g = \sum_{l=1}^{m} b_l \chi_{F_l}$.

Assume that the $E_j$ are p.w. disjoint and the $F_l$ are p.w. disjoint. Then

$$f + g = \sum_{j=1}^{k} a_j \chi_{E_j \setminus (F_1 \cup \cdots \cup F_m)}$$

$$+ \sum_{l=1}^{m} b_l \chi_{F_l \setminus (E_1 \cup \cdots E_k)}$$

$$+ \sum_{j=1}^{k} \sum_{l=1}^{m} (a_j + b_l) \chi_{E_j \cap F_l}.$$ This is simple.

Product is similar.

For scalar multiplication,

$$c \cdot f = c \sum_{j=1}^{k} a_j \chi_{E_j} = \sum_{j=1}^{k} c a_j \chi_{E_j}$$ which is simple.

2. Let $f = \sum_{j=1}^{k} a_j \chi_{E_j}$ and $g = \sum_{l=1}^{m} b_l \chi_{F_l}$.

Then

$$F = \max\{f, g\} = \sum_{j=1}^{k} a_j \chi_{E_j \setminus (F_1 \cup \cdots \cup F_m)}$$

$$+ \sum_{l=1}^{m} b_l \chi_{F_l \setminus (E_1 \cup \cdots \cup E_k)} + \sum_{j=1}^{k} \sum_{l=1}^{m} \max\{a_j, b_l\} \chi_{E_j \cap F_l}.$$

This is simple.

Similarly for G.

4. The hypothesis of LMCT are satisfied. And

$$\lim_{j\to\infty} \int f_j \, d\mu = \lim_{j\to\infty} j = +\infty = \int f \, d\mu .$$

So all is well.

5. In fact $f_j \to 0$ pointwise. So $f \equiv 0$. And

$$\int f \, d\mu = 0 . \qquad (*)$$

But $\int f_j = \frac{1}{j} \cdot (+\infty) = +\infty$.

So $\lim_{j\to\infty} \int f_j = +\infty$. $(**)$.

Note that $(*)$ and $(**)$ are unequal.

7. It is true that $g_j \to g$ a.e. Also $\int g_j \, d\mu = 1$ for each $j$ so $\lim_{j\to\infty} \int g_j \, d\mu = 1$.

But $\int g \, d\mu = \int 0 \, d\mu = 0$. And $0 \neq 1$.

The $g_j$ are <u>not</u> monotone increasing so LMCT does not apply.

On the other hand,

$$\int \liminf g_j \, d\mu = \int 0 \, d\mu = 0$$

while

$$\liminf \int g_j \, d\mu = \liminf 1 = 1$$

and $$0 \le 1.$$

Certainly Fatou's Lemma applies and gives a valid answer.

8. Since the $f_j$ converge uniformly, they also converge pointwise. So $f$ is measurable. It is obvious that $f \ge 0$ a.e. since each $f_j \ge 0$. Now the space has finite measure. So, if $\int f \, d\mu = +\infty$, then $f$ must equal $+\infty$ on a set of positive measure $E$. So, given $M > 0$, $\exists J > 0$ s.t. $j > J \Rightarrow f_j > M$ on $E$. Hence

$$\int f_j \, d\mu \ge M \cdot \mu(E)$$

so that $\int f_j \, d\mu \to +\infty$.

In case $f$ has finite integral, then choose $J$ so large that $j > J \Rightarrow |f_j - f| < 1$.

Then $g = f + 1$ is an upper bound for all $f_j$ with $j > J$. So the dominated convergence theorem applies and

$$\lim_{j\to\infty} \int f_j \, d\mu = \int \lim_{j\to\infty} f_j \, d\mu.$$

10. We see that

$$\int f_j \, d\mu = -\frac{1}{j} \cdot j = -1.$$

$$\int \liminf f_j \, d\mu = \int 0 \, d\mu = 0$$

$$\liminf \int f_j \, d\mu = \liminf(-1) = -1.$$

But $\quad 0 \not\leq -1.$

11. We write

$$\{x \in X : f(x) > 0\} = \bigcup_{j=1}^{\infty} \{x \in X : f(x) > \tfrac{1}{j}\}. \quad (*)$$

Now $\mu\{x \in X : f(x) > \frac{1}{j}\} \leq \int_{\{x \in X : f(x) > 1/j\}} \frac{f(x)}{1/j} \, d\mu(x)$

$$\leq j \int_X f(x) \, d\mu(x).$$

So each of the sets in (*) has finite measure.
Thus $Z$ is $\sigma$-finite.

12. Let $f_j = f \cdot \chi_{[-j,j]}$. Then $f_1 \le f_2 \le \dots$

so LMCT applies and

$$\lim_{j\to\infty} \int f_j \, d\mu = \int \lim f_j \, d\mu = \int f \, d\mu.$$

Let $\varepsilon > 0$. Choose $j$ so large that

$$\left| \int f_j \, d\mu - \int f \, d\mu \right| < \varepsilon.$$

Then
$$\int f \, d\mu \le \int f_j \, d\mu + \varepsilon$$
$$= \int f \cdot \chi_{[-j,j]} \, d\mu + \varepsilon.$$

Thus the result is true with $E = [-j, j]$.