SOLUTIONS TO HW 4

1. \( m \{ x : |f(x)| > \alpha \} \leq \int_{\{x : |f(x)| > \alpha \}} \frac{|f(x)|}{\alpha} \, dx \leq \int_{\mathbb{R}} |f| \, dx < \infty. \)

\( \{ x : f(x) \neq 0 \} = \bigcup_{j=1}^{\infty} \{ x : |f(x)| > \frac{1}{j} \}. \) And each of these sets has finite measure by the same argument as above.

2. It is enough to treat nonnegative \( f. \) We know that there are simple functions \( s_j \) such that \( 0 \leq s_1 \leq s_2 \leq s_3 \leq \ldots \) and \( s_j \rightarrow f \) pointwise. By \( \text{LIM GT}, \)

\[ \int s_j \, dx \rightarrow \int f \, dx. \]

Thus, if \( \varepsilon > 0, \) there is a \( j \) so large that

\[ \int |f - s_j| \, dx = \int f - s_j \, dx < \varepsilon. \]

3. Let \( |g(x)| \leq C \quad \text{for} \quad x. \) Then

\[ \int |f - g| \, dx \leq C \int |f| \, dx < +\infty. \]
5. If it is not the case that $f_1 = f_2$, i.e., then there is a set $E$ of positive measure on which $f_1 \neq f_2$. Then there is a subset $F \subseteq E$ of positive measure and an $\varepsilon > 0$ s.t. either $f_1 > f_2 + \varepsilon$ or $f_2 > f_1 + \varepsilon$ on $F$. Say it's the first. Then

$$\int_{F} f_1 \, du \geq \int_{F} f_2 + \varepsilon \, du \geq \int_{E} f_2 \, du + \varepsilon \cdot m(F).$$

That is a contradiction.

7. Let $X = \mathbb{R}$ and let $f_j(x) = \frac{1}{j}$. Then $f_j \to f = 0$ uniformly. But

$$0 = \int \int f \, du \neq \lim_{j \to \infty} \int f_j \, du = +\infty.$$

8. The example in #7 above will do the trick.

9. Since $\sum_{j=2}^{\infty} \int f_j \, du < +\infty$ we know that
\[
\sum_{j=1}^{\infty} |f_j| \, du < +\infty.
\]

So \( \sum_{j=1}^{\infty} |f_j| \) is finite almost everywhere.

Hence \( \sum_{j=1}^{\infty} f_j \) converges, i.e.,

And

\[
\sum_{j=1}^{\infty} |f_j| \, du \leq \sum_{j=1}^{\infty} |1_f| \, du = \sum_{j=1}^{\infty} |1_f| \, du < +\infty.
\]

10. We have

\[
|\sum_{j=1}^{\infty} |f_j| \, du - \sum_{j=1}^{\infty} |1_f| \, du| = |\sum_{j=1}^{\infty} |f_j| - |1_f| \, du| \leq |\sum_{j=1}^{\infty} |f_j| - |1_f| \, du| \rightarrow 0.
\]

12. We prove this for \( f \) nonnegative. Then it is clear that

\( f_1 \leq f_2 \leq \cdots \)

and \( f_j \rightarrow f \) a.e. So LMCT yields
$S_{fdn} = \lim_{j \to 00} S_{fjdnu}$.

The last assertion also follows from LMGTV.