SOLUTIONS TO HW5

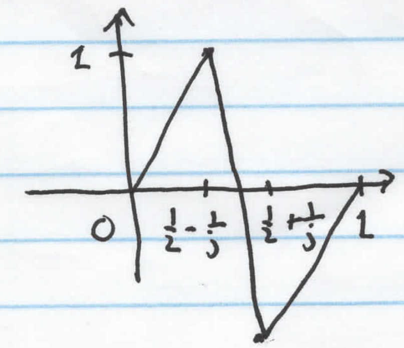
Ch. 5

1. It is clear that $C([0,1])$, the continuous functions on the interval $[0,1]$, is a vector space. To see that it is complete, suppose that $\{f_j\}$ is Cauchy. Then $\{f_j\}$ is Cauchy in the uniform topology; if $\varepsilon > 0$ then there is a $J > 0$ such that $j, k > J$ implies $|f_j(x) - f_k(x)| < \varepsilon$ for all $x \in [0,1]$. But then $\{f_j\}$ converges uniformly. And the limit function will be continuous. So $C[0,1]$ is a Banach space.

3. Define

$$f_j(x) = \begin{cases} \frac{2j}{j-2}x & \text{if } 0 \le x < \frac{1}{2} - \frac{1}{j} \\ -jx + \frac{j}{2} & \text{if } \frac{1}{2} - \frac{1}{j} \le x \le \frac{1}{2} + \frac{1}{j} \\ \frac{2j}{j-2}x - \frac{2j}{j-2} & \text{if } \frac{1}{2} + \frac{1}{j} < x \le 1 \end{cases}$$

for $j = 3, 4, 5, \ldots$.



Then each $f_j \in C[0,1]$.

And $\{f_j\}$ converges in L^1 norm to

$$g(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x-2 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

This g is discontinuous. So <u>not</u> a Banach space.

6. For a continuous function, Lebesgue integral and Riemann integral coincide.

Now

$$\int_1^\infty |f(x)| dx = \lim_{N \to +\infty} \int_1^N \frac{1}{x} dx$$

$$= \lim_{N \to +\infty} \ln x \Big]_1^N = \lim_{N \to +\infty} \ln N = +\infty.$$

So $f \notin L^1([1,\infty))$. But, for $p > 1$,

$$\int_1^\infty |f(x)|^p dx = \lim_{N \to \infty} \int_1^N \frac{1}{x^p} = \lim_{N \to \infty} \frac{1}{-p+1} x^{-p+1} \Big]_1^N$$

$$= \frac{1}{p-1}.$$

So $f \in L^p([1,\infty))$.

9. We have that

$$\int_0^1 |f(x)|^r\,dx = \int_0^1 |f(x)|^r \cdot 1\,dx$$

$$\le \left(\int_0^1 \left(|f(x)|^r\right)^{p/r} dx\right)^{r/p} \cdot \left(\int_0^1 1^{\frac{p}{p-r}}\,dx\right)^{\frac{p-r}{p}}$$

$$= \|f\|_{L^p}^r \cdot 1 = \|f\|_{L^p}^r.$$

So $f \in L^r$.

13. Apply the Minkowski inequality to the measure space consisting of $\mathbb{N}$ equipped with counting measure.

Ch. 6

2. Begin with the interval $[0, 1]$. Remove a middle open interval of length $1/5$. Then remove two intervals of length $1/25$. And so on. The sum of the lengths of all the removed intervals is

$$\frac{1}{5} + \frac{2}{25} + \frac{4}{125} + \cdots$$

$$= \sum_{j=1}^{\infty} \frac{2^{j-1}}{5^j} = \sum_{j=0}^{\infty} \frac{2^j}{5^{j+1}} = \frac{1}{5} \sum_{j=0}^{\infty} \left(\frac{2}{5}\right)^j$$

$$= \frac{1}{5} \cdot \frac{1}{1-\frac{2}{5}} = \frac{1}{5-2} = \frac{1}{3}.$$

Thus the remaining "Cantor-like set" has measure $2/3$.

6. Let E_j be sets with outer measure 0. Let $\varepsilon > 0$. Choose $\{U_k^j\}_{k=1}^{\infty}$ open intervals so that

$$E_j \subseteq \bigcup_{k=1}^{\infty} U_k^j \quad \text{and} \quad \sum_{k=1}^{\infty} \ell(U_k^j) < \frac{\varepsilon}{2^j}.$$

Then $\{U_k^j\}_{j,k=1}^{\infty}$ are open intervals that cover $\bigcup_{j=1}^{\infty} E_j$ and $\sum_{j,k=1}^{\infty} \ell(U_k^j) \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$

Hence $\bigcup_{j=1}^{\infty} E_j$ has outer measure 0.

For each $x \in \mathbb{R}$, the set $S_x = \{x\}$ has outer measure 0. But

$$\bigcup_{x \in \mathbb{R}} S_x = \mathbb{R},$$ and this has outer measure ∞.

9. Let $S \subseteq \mathbb{R}$ be a set. Let $\varepsilon > 0$. Choose intervals U_j so that $\bigcup_{j=1}^{\infty} U_j \supseteq S$ and $\sum_{j=1}^{\infty} \ell(U_j) \leq m^*(S) + \varepsilon$. If $a \in \mathbb{R}$ and

$$\tau_a(S) = \{s + a : s \in S\}.$$

Then the set $\tau_a(U_j)$ are intervals that cover $\tau_a(S)$.

Also $\sum_{j=1}^{\infty} \ell(\tau_a(U_j)) \leq m^*(S) + \varepsilon$.

The same reasoning applies with the roles of S and $\tau_a(S)$ reversed. So

$$m^*(S) = m^*(\tau_a(S)).$$

Ch. 7

1. Let A be any set. We need to see that

$$m^*(A) = m^*(A \cap [0,1]) + m^*(A \setminus [0,1]).$$

If $\{U_j\}$ is a cover of A by open intervals.
Then we can break this up into those intervals $\mathcal{A}$ that intersect $[0,1]$ and the intervals $\mathcal{B}$ that do not. Let $\varepsilon > 0$. If the intervals are all shorter than ε, then it easy to see that the sum of the lengths of the intervals in $\mathcal{A}$ is within 2ε of $m^*(A \cap [0,1])$ and the sum of the lengths of the intervals in $\mathcal{B}$ is within 2ε of $m^*(A \setminus [0,1])$.

4. For each real number r, let

$$S_r = \{q \in \mathbb{Q} : r < q < r+1\}.$$

Then each S_r has measure 0. And the S_r are distinct. There are uncountably many S_r.

6. If $S \subseteq \mathbb{R}$ is a set, define

$$m_*(S) = \sup \ell(K)$$

where $K \subseteq S$, K is compact. Here we define the length ℓ of a compact set K by letting I be an open interval that contains K, calculating the length of $I \setminus K$, and then subtracting that length from the length of I.

There do exist sets for which $m_* \neq m^*$, but they are difficult to construct.

8. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ each be a σ-algebra. Let $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. We check the axioms of a σ-algebra for $\mathcal{A}$:

(i) $\emptyset$ and X both belong to $\mathcal{A}_1$ and both belong to $\mathcal{A}_2$ so they both belong to $\mathcal{A}_1 \cap \mathcal{A}_2$.

(ii) If $E \in \mathcal{A}$ then $E \in \mathcal{A}_1$ so ${}^cE \in \mathcal{A}_1$. Also $E \in \mathcal{A}_2$ so ${}^cE \in \mathcal{A}_2$. It follows that ${}^cE \in \mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}$.

(iii) If E_j are sets in $\mathcal{A}$ then the E_j lie in $\mathcal{A}_1$ so $\bigcup_j E_j \in \mathcal{A}_1$. Also the E_j lie in $\mathcal{A}_2$ so $\bigcup_j E_j \in \mathcal{A}_2$. It follows that $\bigcup_j E_j \in \mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}$. So $\mathcal{A}$ is a σ-algebra.