Math 4121 Spring, 2018
Krantz

Midterm Exam

General Instructions: Write out the solution to each problem on standard 8.5 in. × 11 in. paper. Make your solutions as complete as possible. Leave nothing to the imagination.

You may consult your notes or the textbook or any book in the library. Of course you may talk to me about the exam. But you should *not* talk to each other or to anyone else.

This exam is worth 100 points. It is due in class on Wednesday, April 4, 2018.

(8 points) **1.** Consider the collection $\mathcal{A}$ of all intervals in the real line of the form (a, b), $[a, b)$, (a, ∞), or $[a, \infty)$. Here a can be a real number or $-\infty$. And b can be a real number. Is $\mathcal{A}$ a σ-algebra? Why or why not?

(8 points) **2.** Using the axioms of a σ-algebra, as stated in the text, show that a σ-algebra is closed under countable intersection.

(8 points) **3.** Let $(X, \mathcal{X}, \mu)$ be a measure space, and assume that $\mu(X) = 1$. Let f be a bounded, measurable function on X. Prove that

$$\lim_{p \to +\infty} \|f\|_{L^p} = \|f\|_{L^\infty} .$$

(8 points) **4.** Show that if $(X, \mathcal{X}, \mu)$ is a finite measure space and if $1 \leq p_1 < p_2 \leq +\infty$, then $L^{p_2} \subseteq L^{p_1}$. Then show that this inclusion fails if the measure space is $\mathbb{R}$ equipped with Lebesgue measure.

(10 points) **5.** Consider $\mathbb{R}$ equipped with Lebesgue measure. Let $f_1, f_2, \ldots$ be measurable functions on $\mathbb{R}$ with $f_1(x) \le f_3(x)$ for all x, $f_2(x) \le f_4(x)$ for all x, $f_3(x) \le f_5(x)$ for all x, and so forth. Is it the case that

$$\lim_{j \to +\infty} \int f_j \, d\mu(x) = \int \lim_{j \to +\infty} f_j \, d\mu(x)\,?$$

Either give a proof or a counterexample.

What positive conclusion can you draw from these hypotheses?

(8 points) **6.** Consider $\mathbb{R}$ equipped with Lebesgue measure. Let $f_1, f_2, \ldots$ be measurable functions on $\mathbb{R}$ with $f_1(x) \ge f_2(x) \ge f_3(x) \cdots \ge 0$. Also assume that f_1 is integrable. Show that

$$\lim_{j \to +\infty} \int f_j \, d\mu = \int \lim_{j \to +\infty} f_j \, d\mu\,.$$

(10 points) **7.** Consider $\mathbb{R}$ equipped with Lebesgue measure. For $j = 1, 2, \ldots$, let

$$f_j(x) = \begin{cases} 1 + \sin jx & \text{if} \quad -\pi \le x \le \pi \\ 0 & \text{if} \quad x < -\pi \\ 0 & \text{if} \quad x > \pi\,. \end{cases}$$

What does Fatou's Lemma say about these f_j?

(8 points) **8.** Consider Lebesgue measure on the real line. Let $f \in L^2$. Show that there is a sequence of simple functions s_j so that

$$\lim_{j \to +\infty} \|f - s_j\|_{L^2} = 0\,.$$

(8 points) **9.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that it cannot be the case that $f^{-1}(\{x\})$ is an interval of positive length for every $x \in \mathbb{R}$.

(8 points) **10.** Give an explicit example of a Lebesgue integrable function that is not Riemann integrable. Say precisely why these attributes hold for this function.

(8 points) **11.** Consider Lebesgue measure on the real line. Let f and g be nonnegative integrable functions with

$$f(x) < g(x)$$

for all x. Show that there is a large set E of finite measure so that

$$\int_E f \, d\mu \leq \int_E g \, d\mu \, .$$

(8 points) **12.** Use Lebesgue measure on the real line. Fix a number $1 < p_0 < \infty$. Give an example of a function that is in L^{p_0} but in no other L^p space.

(10 points) **EXTRA CREDIT:** Consider the real line equipped with Lebesgue measure. Focus on the space $X = L^2$. Show that if $X = \cup_j E_j$, then it cannot be the case that, for each j, the closure of E_j contains no open ball.