SOLUTIONS TO MIDTERM

2. (a) The set \((0, 0) = \emptyset\) lies in \(A\).
   The set \((-\infty, \infty) = \mathbb{R}\) lies in \(A\).

(b) The set \((0, 1)\) lies in \(A\),
   but the complement of this set is
   \((-\infty, 0] \cup [1, \infty)\) and that does not lie
   in \(A\),
   So \(A\) is not a \(\sigma\)-algebra.

2. If \(E_1, E_2, \ldots\) lie in the \(\sigma\)-algebra, then
   \[
   \bigcap_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} E_j \right)'
   \]
   by de Morgan's Law. And the \(\sigma\)-algebra is closed
   under complementation and countable union.
   So \(\bigcap_{j=1}^{\infty} E_j\) lies in the \(\sigma\)-algebra.

3. Let \(M = \|f\|_{L^\infty}\). Then
   \[
   \|f\|_{L^p} = \left( \int |f|^p \, dx \right)^{1/p} \leq \left( \int M^p \, dx \right)^{1/p} = M,
   \]
   So linearly \(\|f\|_{L^p} \leq M\).

   Now let \(\varepsilon > 0\). Choose a set \(E\) of positive
   measure so that \(|f| \geq M - \varepsilon\) on \(E\).
Then

\[ \|f\|_{L^p} \geq \left( \int_{-\infty}^{\infty} (M - \varepsilon)^p \, du \right)^{1/p} = (M - \varepsilon) \mu(E)^{1/p}. \]

Thus

\[ \liminf_{p \to \infty} \|f\|_{L^p} \geq M - \varepsilon. \]

Since this is true for all \( \varepsilon > 0 \), we see that

\[ \liminf_{p \to \infty} \|f\|_{L^p} \geq M. \]

And

\[ \limsup_{p \to \infty} \|f\|_{L^p} \leq M. \]

Hence

\[ \lim_{p \to \infty} \|f\|_{L^p} = M = \|f\|_{L^\infty}. \]

4. Let \( f \in L^{p_2}. \) Then

\[ \|f\|_{L^{p_2}} = \left( \int |f|^{p_2} \, du \right)^{1/p_2} = \left( \int |f|^{p_2} \cdot 1 \, du \right)^{1/p_2} \]

(Hölder's)

\[ \leq \left( \int |f|^{p_2} \, du \right)^{1/p_2} \cdot \left( \int 1^{p_2/p_2} \, du \right)^{p_2/p_2} \]

\[ \leq C \cdot \|f\|_{L^{p_2}}. \]

So \( f \in L^{p_1}. \)
On the real line, consider the function
\[ f(x) = \begin{cases} \frac{1}{x^{1/3}} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases} \]

Then
\[ \int |f|^2 \, du = \int_1^\infty \frac{1}{x^{2/3}} \, dx < \infty \]
because \( p_2 / p_1 > 1 \).

But
\[ \int |f|^1 \, du = \int_1^\infty \frac{1}{x} \, dx = +\infty \]
so \( f \notin L^1 \).

5. Let
\[ f_1(x) = f_3(x) = f_5(x) = \ldots = 1 \text{ for } 0 \leq x < 1, \quad 0 \text{ otherwise}, \]
\[ f_2(x) = f_4(x) = f_6(x) = \ldots = 2 \text{ for } 0 \leq x < 1, \quad 0 \text{ otherwise}. \]

Then
\[ \int f_{2j-1} \, du = 1 \]
\[ \int f_{2j} \, du = 2 \quad \text{for } j = 1, 2, \ldots \]

So
\[ \lim_{j \to \infty} \int f_j \, du \text{ does not exist} \]
and
\[ \lim_{j \to \infty} \int f_j \, du \text{ does not exist}. \]
For a positive result, let
\[ g_1 = f_1 + f_2 \]
\[ g_2 = f_3 + f_4 \]
\[ g_3 = f_5 + f_6 \]

etc.

Then \( g_1 \leq g_2 \leq g_3 \leq \ldots \). So LMC T applies and
\[ \lim_{j \to \infty} \int g_j \, du = \int \lim_{j \to \infty} g_j \, du \]

6. Clearly \( \lim_{j \to \infty} f_j(x) \) exists. Also LMC T applies

with \( g = f_1 \). So
\[ \lim_{j \to \infty} \int f_j \, du = \int \lim_{j \to \infty} f_j \, du \]

7. We see that
\[ \lim_{j \to \infty} \inf f_j(x) = 0 \]

And\[ \lim_{j \to \infty} \left( \int f_j \, du \right) = 2\pi \]

So \( 0 \leq 2\pi \) is consistent with Fatou's Lemma.
8. We may as well assume that \( f \geq 0 \). Then we know that there are simple functions \( s_j \) such that \( s_1 \leq s_2 \leq \ldots \leq f \) and \( s_j \to f \) pointwise. Then L,MCT tells us that

\[
\lim_{j \to \infty} \int s_j^2 \, dx = \int \lim_{j \to \infty} s_j^2 \, dx = \int f^2 \, dx.
\]

Also

\[
\lim_{j \to \infty} \int s_j \, dx = \int \lim_{j \to \infty} s_j \, dx = \int f \, dx.
\]

Hence

\[
\lim_{j \to \infty} \int (f - s_j)^2 \, dx = \lim_{j \to \infty} \int (f^2 - 2s_jf + s_j^2) \, dx
\]

\[
= \lim_{j \to \infty} \int f^2 \, dx - 2 \int s_j \, dx + \int s_j^2 \, dx
\]

\[
= \int f^2 \, dx - 2 \int s_j \, dx + \int s_j^2 \, dx
\]

\[
= \int s_j^2 \, dx - 2 \int s_j \, dx + \int f^2 \, dx - \int f^2 \, dx = 0.
\]

9. Each \( f^{-1}(\mathbb{R} \times 0) \) is disjoint from \( f^{-1}(\mathbb{R} \times 0) \) if \( x \neq x' \). And each \( f^{-1}(\mathbb{R} \times 0) \) contains a distinct rational number. So there could only be countably many of them. Contradiction.
20. Let $F(x) = \chi_\mathbb{Q}(x)$. Then $F$ is Lebesgue integrable and 
\[ \int f \, du = 0. \]

But $f$ is not Riemann integrable because it is discontinuous at every point (and a Riemann integrable function has discontinuities that form a set of measure 0).

11. Write

\[ g - f = (g - f)^+ - (g - f)^-. \]

First we treat $$(g - f)^+.$$ For $j \in \mathbb{N}$, set $h_j = \frac{1}{j} - (g - f)^+.$

Then $h_1 \leq h_2 \leq \ldots$. So LMCT applies and

\[ \lim_{j \to \infty} \int h_j \, du = \int (g - f)^+ \, du > 0. \]

Hence, for $j$ large,

\[ \int h_j \, du > 0 \]

or

\[ \int (g - f)^+ \, du > 0. \]

Of course, by hypothesis, $\int (g - f)^- \, du < \int (g - f)^+ \, du.$ So, for some $j$ large enough, \( \int (g - f)^+ - (g - f)^- \, du > 0. \)
22. Let
\[ f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{x^{1/\rho} \log x^{1/\rho}} & \text{if } 0 < x < \frac{1}{2} \text{ or } 2 < x < \infty \\ 0 & \text{if } \frac{1}{2} \leq x \leq 2 \end{cases} \]
Then
\[ |f(x)|^p = \frac{1}{x (\log x)^{2}} \]
i.e. \(0 < x < \frac{1}{2} \text{ or } 2 < x < \infty\).

Hence
\[
\int |f(x)|^p \, dx = \int_{0}^{\infty} \frac{1}{x \log^2 x} \, dx = \int_{0}^{\frac{1}{2}} + \int_{2}^{\infty} \frac{1}{x \log^2 x} \, dx
\]
\[= -\frac{1}{\log x} \bigg|_{0}^{\frac{1}{2}} + \frac{1}{\log x} \bigg|_{2}^{\infty}
\]
\[= \frac{1}{\log 2} + \frac{1}{\log 2} = \frac{2}{\log 2}
\]
If \( p' > p \) then, for \( 0 < x < 1/2 \),
\[ |f(x)|^{p'} = \frac{1}{x^{1/\rho} \log x^{1/\rho}^{2p'/p} > \frac{1}{x^{\rho'/\rho - \varepsilon}} \}
\]
where \( \varepsilon \) is chosen so that \( \frac{\rho'}{\rho} - \varepsilon > 1 \). Then
\[ |f(x)|^{p'} \text{ is not integrable at the origin.}
\]
If \( p' < p \) then, for \( x > 2 \)
\[ |f(x)|^{p'} = \frac{1}{x^{1/\rho} \log x^{1/\rho}^{2p'/p} > \frac{1}{x^{\rho'/\rho + \varepsilon}} \}
\]
where \( \varepsilon \) is chosen so that \( \frac{\rho'}{p} + \varepsilon < 1 \).
Then \(|f(x)|^p\) is not integrable at \(+\infty\).

**EXTRA CREDIT PROBLEM:** This is actually an instance of the Baire Category Theorem. See


for statement, discussion, and proof. This s.t. actually states and proves the contrapositive.