Math 4121 Spring, 2021
Krantz

Practice Exam for First Midterm

1. Consider the collection $\mathcal{A}$ of all intervals in the real line of the form (a, b), $[a, b)$ where a and b are finite real numbers. Is $\mathcal{A}$ a σ-algebra? Why or why not?

2. Give an example to show that a σ-algebra may *not* be closed under uncountable union.

3. Let $(X, \mathcal{X}, \mu)$ be a measure space, and assume that $\mu(X) = 1$. If $p_2 > p_1$ then show that

$$\|f\|_{L^{p_1}} \leq \|f\|_{L^{p_2}}$$

for any $f \in L^{p_2}$.

4. Work on the real line with Lebesgue measure. Let $p_1 < p_2 < p_3$. Show that

$$L^{p_2} \supseteq L^{p_1} \cap L^{p_3} .$$

5. Work on the real line with Lebesgue measure. Give an example of integrable functions $0 \leq f_1 \leq f_2 \leq \cdots$ so that $\lim_{j\to\infty} \int f_j dx = +\infty$.

6. Prove Weyl's lemma: If α is an irrational number and we consider the numbers $[n\alpha]$, which are the fractional parts of $n\alpha$ for n a positive integer, then $[n\alpha]$ is dense in the interval $[0, 1]$.

7. Consider $\mathbb{R}$ equipped with Lebesgue measure. Let $f_j(x) = 1 + \cos jx$ for $j = 1, 2, \ldots$. What does Fatou's Lemma say about these f_j?

8. Consider Lebesgue measure on the real line. Let $f \in L^2$, $f \geq 0$. Show that there is a sequence of functions $g_j \in L^4$ so that $g_j \to f$ in the L^2 topology.

9. Work on the real line with Lebesgue measure. Let $f \in L^1$. Prove that there is a sequence g_j of continuous functions with compact support so that $g_j \to f$ in the L^1 topology.

10. Give an explicit example of a bounded function that is not Lebesgue integrable.

11. Consider Lebesgue measure on the real line. Let f be a square integrable function. Show that f can be written in the form $g + h$, where $g \in L^1$ and $h \in L^4$.

12. Use Lebesgue measure on the real line. Fix numbers $1 < p_0 < p_1 < \infty$. Give an example of a function that is in L^{p_0} but not in L^{p_1}. Give an example of a function that is in L^{p_1} but not in L^{p_0}.