DATE KRANTZ

PURPOSE Spring, 2021

SOLUTIONS TO PRACTICE MIDTERM 1

CUE COLUMN

NOTES

1. Note that

$$\bigcup_{j=1}^{\infty} [j, j+1) = [0, \infty),$$

so this collection of intervals is not countably additive. Hence not a σ-algebra.

2. The Lebesgue non-measurable set is the uncountable union of singletons. Each singleton is Borel, but the Lebesgue set is not.

3. We calculate that

$$\int_X |f(x)|^{p_1} d\mu(x) = \int_X \left(|f(x)|^{p_1}\right) \cdot 1 \, d\mu(x)$$

(Hölder)

$$\leq \int_X \left(\left(|f(x)|^{p_1}\right)^{p_2/p_1} d\mu(x)\right)^{p_1/p_2} \cdot$$

$$\int_X 1^{p_2/(p_2-p_1)} d\mu(x)^{\frac{p_2-p_1}{p_2}}$$

$$= \int_X |f(x)|^{p_2} d\mu(x)^{p_1/p_2}$$

Hence $\int_X |f(x)|^{p_1} d\mu(x)^{1/p_1} \leq \int_X |f(x)|^{p_2} d\mu(x)^{1/p_2}$

SUMMARY

$$\|f\|_{p_1} \leq \|f\|_{p_2}.$$

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4. Let $f \in L^{p_1} \cap L^{p_3}$. Then set

$$E = \{x : |f(x)| \le 1\},$$

$$F = \{x : |f(x)| > 1\}$$

Then $\int_E |f(x)|^{p_1} dx < \infty$

hence $\int_E |f(x)|^{p_2} dx \le \int_E |f(x)|^{p_1} dx < \infty$.

Also $\int_F |f(x)|^{p_3} dx < \infty$

hence $\int_F |f(x)|^{p_2} dx \le \int_F |f(x)|^{p_3} dx < \infty$.

In sum,

$$\int_{\mathbb{R}} |f(x)|^{p_2} dx = \int_E |f(x)|^{p_2} dx + \int_F |f(x)|^{p_2} dx$$

$$< \infty.$$

So $f \in L^{p_2}$.

5. Let $f_j(x) = e^{-x^2/j}$, $j = 1, 2, \ldots$. Then

$f_1 \le f_2 \le \ldots$. Each f_j is integrable.

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But $\lim_{j \to \infty} f_j(x) \equiv 1$, which is not integrable.

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6. The numbers $[n\alpha]$ are a bounded sequence, so there is a convergent subsequence $[n_j\alpha]$. Let $\varepsilon > 0$. Choose J so large that $j, k > N$

$\Rightarrow |[n_j\alpha] - [n_k\alpha]| < \varepsilon$. Hence

$|[(n_j - n_k)\alpha]| < \varepsilon$.

But then

$[(n_j - n_k)\alpha], [2(n_j - n_k)\alpha], \ldots [[1/\varepsilon](n_j - n_k)\alpha]$

form an ε-net in $[0, 1]$.

Since this construction is valid for all $\varepsilon > 0$, we see that the set of $[n\alpha]$ is dense in $[0, 1]$.

7. Using Weyl's lemma (problem 6); it can be checked that

$$\liminf_{j\to\infty} f_j(x) = 0 \quad \text{a.e.}$$

But clearly $\int f_j(x)\,dx = \infty \quad \forall j$. So

$$0 = \int \liminf f_j(x)\,dx < \liminf_{j\to\infty} \int f_j(x)\,dx = \infty.$$

Hence Fatou is valid.

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8. Let

$$g_j(x) = \begin{cases} f(x) & \text{if } 0 \le f(x) \le j \\ j & \text{if } f(x) > j. \end{cases}$$

Then $g_j \in L^4$, $g_1 \le g_2 \le \ldots$, and $g_j \to f$. Furthermore, $|f-g_j|^2 \le |f|^2$, so by Lebesgue Dominated Convergence,

$$\int |f-g_j|^2 dx \to 0.$$

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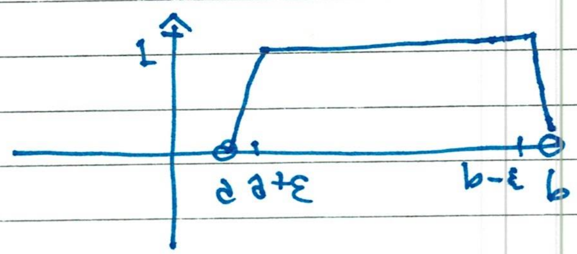
9. First let us treat the case where f is a characteristic function. So

$$f(x) = \chi_E(x).$$

If E is an interval (a, b), then we may approximate f by a continuous function like this:



Similarly if E is a finite union of intervals. But then the complement of a finite union of intervals may be approximated by subtraction.

An infinite union of intervals may be approximated because an infinite sum is the limit of partial sums. Likewise for the complement.

So we can handle the case that E is a Borel set. A Lebesgue measurable set differs from a Borel set by a set of measure 0. So Lebesgue measurable sets are no problem.

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Finally, we can obviously handle simple functions. And f is the L^1 limit of simple functions.

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10. Let E be the Lebesgue non-measurable set. Let $f(x) = \chi_E$. Then f is not Lebesgue integrable.

11. Let

$$E = \{x : |f(x)| \leq 1\}$$

$$F = \{x : |f(x)| > 1\}.$$

Set $g = f \cdot \chi_F$

$$h = f \cdot \chi_E.$$

Then $h \in L^4$, $g \in L^1$, and $f = g + h$.

12. Let

$$f(x) = \frac{1}{x^{1/p_0}} \cdot \chi_{[1,\infty)}.$$

Then $f \in L^{p_1}$ but $f \notin L^{p_0}$.

Let $g(x) = \frac{1}{x^{1/p_1}} \cdot \chi_{(0,1)}$.

Then $f \in L^{p_0}$ but $f \notin L^{p_1}$.

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