Math 4122

SOLUTIONS TO MIDTERM 1

1. (a) The set $(0,0) = \emptyset$ lies in $\mathcal{A}$.
   The set $(-\infty,\infty) = \mathbb{R}$ lies in $\mathcal{A}$.

   (b) The set $(0,1)$ lies in $\mathcal{A}$.
   But the complement of this set is $(-\infty, 0] \cup [1,\infty)$ and that does <u>not</u> lie in $\mathcal{A}$.

So $\mathcal{A}$ is <u>not</u> a σ-algebra.

2. If $E_1, E_2, \ldots$ lie in the σ-algebra, then

$$\bigcap_{j=1}^{\infty} E_j = {}^c\left(\bigcup_{j=1}^{\infty} {}^cE_j\right)$$

by de Morgan's Law. And the σ-algebra is closed under complementation and countable union.
So $\bigcap_{j=1}^{\infty} E_j$ lies in the σ-algebra.

3. Let $M = \|f\|_{L^\infty}$. Then

$\|f\|_{L^p} = \left(\int |f|^p \, d\mu\right)^{1/p} \le \left(\int M^p \, d\mu\right)^{1/p} = M$. So $\limsup_{p\to\infty} \|f\|_{L^p} \le M$.

Now let $\varepsilon > 0$. Choose a set E of positive measure so that $|f| \ge M - \varepsilon$ on E.

Then

$$\|f\|_{L^p} \geq \left(\int_E (M-\varepsilon)^p \, d\mu\right)^{1/p} = (M-\varepsilon)\,\mu(E)^{1/p}.$$

Thus $\liminf_{p\to\infty} \|f\|_{L^p} \geq M-\varepsilon.$

Since this is true $\forall \varepsilon > 0$, we see that

$$\liminf_{p\to\infty} \|f\|_{L^p} \geq M.$$

And $\limsup_{p\to\infty} \|f\|_{L^p} \leq M.$

Hence $\lim_{p\to\infty} \|f\|_{L^p} = M = \|f\|_{L^\infty}$

4. Let $f \in L^{p_2}$. Then

$$\|f\|_{L^{p_1}} = \left(\int |f|^{p_1} \, d\mu\right)^{1/p_1} = \left(\int |f|^{p_1} \cdot 1 \, d\mu\right)^{1/p_1}$$

(Hölder)

$$\leq \left(\int |f|^{p_2} \, d\mu\right)^{1/p_2} \cdot \left(\int 1^{\frac{p_2}{p_2-p_1}} \, d\mu\right)^{\frac{p_2-p_1}{p_1 p_2}}$$

$$\leq C \cdot \|f\|_{L^{p_2}}.$$

So $f \in L^{p_1}$.

On the real line, consider the function

$$f(x) = \begin{cases} \dfrac{1}{x^{1/p_1}} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$

Then

$$\int |f|^{p_2}\, d\mu = \int_1^\infty \frac{1}{x^{p_2/p_1}}\, dx < \infty$$

because $p_2/p_1 > 1$.

But

$$\int |f|^{p_1} d\mu = \int_1^\infty \frac{1}{x}\, dx = +\infty$$

so $f \notin L^{p_1}$.

5. Let

$$f_1(x) = f_3(x) = f_5(x) = \cdots \equiv 1 \text{ for } 0 \leq x \leq 1, \ 0 \text{ otherwise}$$
$$f_2(x) = f_4(x) = f_6(x) = \cdots \equiv 2 \text{ for } 0 \leq x \leq 1, \ 0 \text{ otherwise}$$

Then

$$\int f_{2j-1}\, d\mu = 1$$
$$\int f_{2j}\, d\mu = 2 \qquad \text{for } j = 1, 2, \ldots$$

So $\lim_{j\to\infty} \int f_j\, d\mu$ does not exist

and $\int \lim_{j\to\infty} f_j\, d\mu$ does not exist.

For a positive result, let

$$g_1 = f_1 + f_2$$

$$g_2 = f_3 + f_4$$

$$g_3 = f_5 + f_6$$

etc.

Then $g_1 \le g_2 \le g_3 \le \cdots$. So LMCT applies and

$$\lim_{j\to\infty} \int g_j \, d\mu = \int \lim_{j\to\infty} g_j \, d\mu$$

6. Clearly $\lim_{j\to\infty} f_j(x)$ exists. Also LDCT applies with $g = f_1$. So

$$\lim_{j\to\infty} \int f_j \, d\mu = \int \lim_{j\to\infty} f_j \, d\mu.$$

7. It can be checked that $\liminf_{j\to\infty} f_j(x) = 0$.

But $\liminf_{j\to\infty} \int f_j(x)\,dx = \infty$. So $0 \le \infty$. is consistent with Fatou's lemma.

SUMMARY

8. We may as well assume that $f \geq 0$. Then we know that there are simple functions s_j such that $s_1 \leq s_2 \leq \cdots \leq f$ and $s_j \to f$ pointwise. Then LMCT tells us that

$$\lim_{j\to\infty} \int s_j^2 \, d\mu = \int \lim_{j\to\infty} s_j^2 \, d\mu = \int f^2 \, d\mu .$$

Also $$\lim_{j\to\infty} \int f s_j \, d\mu = \int \lim_{j\to\infty} f s_j \, d\mu = \int f^2 \, d\mu .$$

Hence

$$\lim_{j\to\infty} \int |f - s_j|^2 \, d\mu = \lim_{j\to\infty} \int (f-s_j)^2 \, d\mu$$

$$= \lim_{j\to\infty} \int f^2 - 2 f s_j + s_j^2 \, d\mu$$

$$= \int f^2 \, d\mu - 2 \lim_{j\to\infty} \int f s_j \, d\mu + \lim_{j\to\infty} \int s_j^2 \, d\mu$$

$$= \int f^2 \, d\mu - 2 \int f^2 \, d\mu + \int f^2 \, d\mu = 0 .$$

9. If each $f^{-1}(\{x\})$ is an interval of positive length, then each contains a rational. But there are uncountably many of those, and they are pairwise disjoint. Contradiction.

10. Let $f(x) = \chi_{\mathbb{Q}}(x)$. Then f is Lebesgue integrable and
$$\int f \, d\mu = 0.$$

But f is not Riemann integrable because it is discontinuous at every point (and a Riemann integrable function has discontinuities that form a set of measure 0).

11. Write
$$g - f = (g-f)^+ - (g-f)^-.$$

First we treat $(g-f)^+$. For $j \in \mathbb{N}$, set $h_j = \chi_{[-j,j]} \cdot (g-f)^+$.

Then $h_1 \le h_2 \le \ldots$. So LMCT applies and
$$\lim_{j \to \infty} \int h_j \, d\mu = \int (g-f)^+ d\mu > 0.$$

Hence, for j large,
$$\int h_j \, d\mu > 0$$
or
$$\int_{[-j,j]} (g-f)^+ d\mu > 0.$$

Of course, by hypothesis, $\int (g-f)^- d\mu < \int (g-f)^+ d\mu$. So, for j large enough, $\int_{[-j,j]} (g-f)^+ - (g-f)^- d\mu > 0$.

12. Let

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{1}{x^{1/p_0}|\log x|^{2/p_0}} & \text{if } 0 < x < \frac{1}{2} \text{ or } 2 < x < \infty \\ 0 & \text{if } \frac{1}{2} \le x \le 2 \end{cases}$$

Then $|f(x)|^p = \dfrac{1}{x|\log x|^2}$ if $0 < x < \frac{1}{2}$ or $2 < x < \infty$.

Hence

$$\int |f(x)|^{p_0} dx = \int_0^{1/2} \frac{1}{x \log^2 x} dx + \int_2^{\infty} \frac{1}{x \log^2 x} dx$$

$$= \frac{-1}{\log x}\Big]_0^{1/2} + \frac{-1}{\log x}\Big]_2^{\infty}$$

$$= \frac{1}{\log 2} + \frac{1}{\log 2} = \frac{2}{\log 2}$$

If $p' > p_0$ then, for $0 < x < 1/2$,

$$|f(x)|^{p'} = \frac{1}{x^{p'/p_0}|\log x|^{2p'/p_0}} > \frac{1}{x^{p'/p_0 - \varepsilon}},$$

where ε is chosen so that $\frac{p'}{p_0} - \varepsilon > 1$. Then $|f(x)|^{p'}$ is not integrable at the origin.

If $p' < p_0$ then, for $x > 2$,

$$|f(x)|^{p'} = \frac{1}{x^{p'/p_0}|\log x|^{2p'/p_0}} > \frac{1}{x^{p'/p_0 + \varepsilon}}$$

where ε is chosen so that $\frac{p'}{p_0} + \varepsilon < 1$.

So $|f(x)|^{p'}$ is not integrable at ∞.

EXTRA CREDIT! This is a direct application of the Baire category theorem (look it up!).