Math 4121 | Spring, 2021
Krantz

Second Midterm Exam

General Instructions: Write out the solution to each problem on standard 8.5 in. $\times$ 11 in. paper. Only write one problem per page. Make your solutions as complete as possible. Leave nothing to the imagination.

You will upload your solutions to `CrowdMark` so that they can be graded.

You may consult your notes or the textbook or any book in the library. Of course you may talk to me about the exam. But you should *not* talk to each other or to anyone else.

You will have Monday, May 3 through Wednesday, May 5 to do the exam.

This exam is worth 100 points.

(8 points) **1.** What is the outer measure m^* of the Cantor ternary set? Do an *explicit* calculation to answer this question!!

(8 points) **2.** Suppose that E is a Lebesgue measurable set such that $m^*(E) = 0$. What can you say about the Lebesgue measure of E?

(10 points) **3.** Prove that the interval $[0, 1]$ is Lebesgue measurable. You must use the *rigorous* definition of "measurable" to do this problem!!

(8 points) **4.** Prove that there are uncountably many distinct sets of Lebesgue measure 1. You should give an explicit description of each set.

(8 points) **5.** Suppose that P is a positive set with respect to a signed measure μ. Let $E \subseteq P$ be measurable. Prove that E is also a positive set with respect to μ.

(8 points) **6.** Show that any countable subset of $\mathbb{R}$ has Lebesgue measure 0.

(8 points) **7.** Give an example of an uncountable subset of $\mathbb{R}$ that has Lebesgue measure 0.

(10 points) **8.** Use Lebesgue measure λ. Let f be a Lebesgue integrable function on the real line. Let $\epsilon > 0$. Prove that there exists a continuous function φ such that $\|f - \varphi\|_{L^1} = \int |f(x) - \varphi(x)| \, d\lambda(x) < \epsilon$.

(8 points) **9.** Work with Lebesgue measure λ on the real line. Fix an integrable function φ. Consider the linear operator

$$Tf(x) \equiv \int f(x-t)\varphi(t)\, d\lambda(t) \, .$$

Prove that, if f is integrable, then Tf is integrable. Prove that the operator T is "translation invariant" in a suitable sense.

(10 points) **10.** Set $g_j(x) = j \cdot \chi_{[1/j, 2/j]}(x)$ for $j = 1, 2, \ldots$. Prove that the sequence $\{g_j\}$ converges pointwise at every point to the identically 0 function. Prove that the sequence does *not* converge uniformly. Prove that the sequence does *not* converges in L^2. Finally prove that the sequence *does* converge in measure.

(8 points) **11.** Use the ideas from our study of product measures to prove the following. Let $a_{j,k} \geq 0$ for $j, k \in \mathbb{N}$. Then

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k} \, .$$

(6 points) **12.** Let S be a Lebesgue non-measurable set in $\mathbb{R}$. Let N be a null set (that is, a set of measure 0). Is $S \cup N$ measurable or non-measurable? Or neither? Give an *explicit* reason for your answer.