Math 4121 Spring, 2021
Krantz

Practice Second Midterm Exam

1. What is the outer measure m^* of the set of rational numbers?

2. Show that the countable union of sets with outer measure 0 still has outer measure 0.

3. Prove that the set of rational numbers strictly between 0 and 1 is Lebesgue measurable.

4. Prove that the cardinality of the collection of sets of reals with Lebesgue measure 0 is greater than the cardinality of the real numbers.

5. Suppose that N is a negative set with respect to a signed measure μ. Let $E \subseteq N$ be measurable. Prove that E is also a negative set with respect to μ.

6. Show that there is a function on the real line that is Lebesgue measurable but not Borel measurable.

7. Let φ be a fixed integrable function on the reals. Define

$$Tf(x) = \int f(x-t)\varphi(t)\, dt\,.$$

Prove that, if f is square integrable, then Tf is square integrable.

8. Use Lebesgue measure λ. Let f be a Lebesgue integrable function on the real line. Let $\epsilon > 0$. Prove that there exists a piecewise linear function φ such that $\|f - \varphi\|_{L^1} = \int |f(x) - \varphi(x)| \, d\lambda(x) < \epsilon$.

9. Let f be an L^4 function on $\mathbb{R}$. Show that f can be written in the form $f = g + h$, where $g \in L^2$ and $h \in L^6$.

10. Let $1 \leq p < \infty$. Set $g_j(x) = j^{-1/p} \cdot \chi_{[0,j]}(x)$ for $j = 1, 2, \ldots$. Show that g_j converges uniformly to the identically 0 function. Show that the sequence g_j does *not* converge in L^p. But prove that the sequence g_j converges in measure.

11. Let $a_{j,k}$ for $j, k \in \mathbb{N}$ be defined by $a_{j,j} = +1$ for each j, $a_{j,j+1} = -1$ for each j, and $a_{j,k} = 0$ if either $j \neq k$ or $k \neq j + 1$. Show that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k} = 0$$

while

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k} = 1 \, .$$

Explain why this shows that the hypothesis of integrabity in Fubini's theorem is really needed.

12. We know that any set of positive measure contains a non-measurable set. But this is not true for sets of measure zero. Explain why not.