DATE Math 4121 (2)

PURPOSE Spring, 2021

SOLUTIONS TO PRACTICE MIDTERM 2

CUE COLUMN

NOTES

1\. Let $\{q_j\}$ be an enumeration of the rational numbers. Let $\varepsilon > 0$. Set

$$I_j = (q_j - \varepsilon/2^{j+1}, q_j + \varepsilon/2^{j+1}).$$

Then $\{q_j\} \subseteq \bigcup_j I_j$. And

$$\sum_{j=1}^{\infty} \ell(I_j) \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} = \varepsilon.$$

So the outer measure of $\{q_j\}$ is $\le \varepsilon$ for every $\varepsilon > 0$. Hence it equals 0.

2\. Let O_j have outer measure 0 for $j = 1, 2, \ldots$. Let $\mathcal{U}_j$ be a collection of open intervals I^j_k which cover O_j and $\sum_k \ell(I^j_k) \le \varepsilon/2^{j+1}$.

Then $\bigcup_j \mathcal{U}_j$ covers $\bigcup_j O_j$ and

$$\sum_{j,k} \ell(I^j_k) \le \sum_j \frac{\varepsilon}{2^{j+1}} = \varepsilon.$$

Hence the outer measure of $\bigcup_j O_j$ is 0.

3\. We use Carathéodory's criterion. Let X be the set of rationals between 0 and 1. Let A be a set of finite outer measure. Then $A \setminus X \subseteq A$ so $m^*(A \setminus X) \le m^*(A)$.

SUMMARY

Also $A \cap X$ is countable so $m^*(A \cap X) = 0$. Hence $m^*(A) \ge m^*(A \cap X) + m^*(A \setminus X)$.

Thus X is measurable.

DATE

PURPOSE

CUE COLUMN

NOTES

4. Let C be the Cantor ternary set. So C is uncountable, so has the cardinality of the reals. The power set $P(C)$ has greater cardinality than the reals, and each element of $P(C)$ has measure 0.

5. Let S be any element of $\mathcal{X}$. Then $\mu(S \cap E) = \mu((S \cap E) \cap N)$ is ≤ 0. Hence E is negative.

6. Let X be a set that is Lebesgue measurable but not Borel. Then χ_{X} is Lebesgue measurable but not Borel.

7. Let g be square integrable. Then

$$|\langle Tf, g\rangle| = \left|\int Tf(x)\, g(x)\, dx\right|$$

$$\leq \int \int |f(x-t)||\varphi(t)|\,dt\,|g(x)|\,dx$$

$$= \int |f(x-t)\, g(x)|\,dx \int |\varphi(t)|\,dt$$

$$\leq \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^1}.$$

SUMMARY

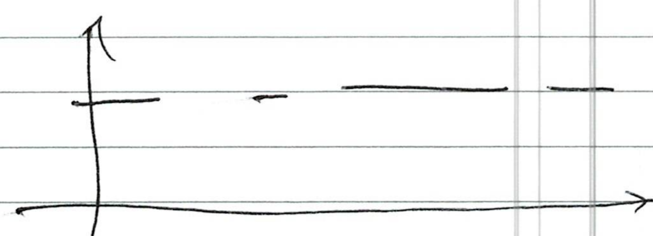
By the Riest representation theorem, $Tf \in L^2$.

CUE COLUMN

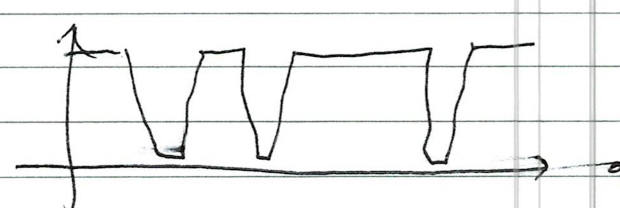
NOTES

8, We know that f can be approximated in L^1 by a simple functions. And each characteristic function in s can be approximated by the characteristic function of a Borel set. And each characteristic function of a Borel set can be approximated by the characteristic function of some intervals.

This in turn



can be approximated by a piecewise linear function



SUMMARY

CUE COLUMN NOTES

9. Write

$$f = f\chi_{\{x : |f(x)| \le 1\}} + f\chi_{\{x : |f(x)| > 1\}}$$

$$\equiv h + g.$$

Then $\int |h|^6 dx = \int_{\{x : |f(x)| \le 1\}} |f|^6 dx \le \int_{\{|f| \le 1\}} |f|^4 dx$

$$\le \|f\|_{L^4}^4.$$

Also

$$\int |g|^2 dx = \int_{\{x : |f(x)| > 1\}} |f|^2 dx \le \int_{\{|f| > 1\}} |f|^4 dx$$

$$\le \|f\|_{L^4}^4.$$

10. Since $|g_j(x)| \le j^{-1/p}$ $\forall j$, $\forall x$, we see that $g_j \to 0$ uniformly. But

$$\int |g_j - 0|^p dx = \int_0^j (j^{-1/p})^p dx = \int_0^j \frac{1}{j} dx = 1.$$

So $\{g_j\}$ does <u>not</u> converge to 0 in L^p.

Finally, for $\alpha > 0$, $j > \alpha + 1$,

SUMMARY

$\lambda\{x : |g_j(x) - 0| > \alpha\} = 0$. So $g \to 0$ in measures

DATE

PURPOSE

CUE COLUMN

NOTES

11. Of course

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k} = \sum_{j=1}^{\infty}(1+-1) = 0$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k} = 1 + \sum_{k=2}^{\infty}\sum_{j=1}^{\infty} a_{j,k}$$

$$= 1 + \sum_{k=2}^{\infty} 0 = 1.$$

Thinking in terms of counting measure on $\mathbb{N}\times\mathbb{N}$, $|a_{j,k}|$ is not integrable because it is not summable. And, as we have seen, Fubini fails.

12. Any subset of a set of measure zero is itself measurable and has measure zero.

SUMMARY