DATE Spring, 2021

PURPOSE Math 4121

SOLUTIONS TO MIDTERM 2

CUE COLUMN

NOTES

1. At the j^{th} step in the construction of the Cantor set we have a compact set S_j consisting of 2^j pairwise disjoint intervals having total length $2^j/3^j$. Thus it is easy to cover S_j by 2^j pairwise disjoint open intervals of length $(2/3)^j + \frac{1}{100} \cdot 3^{-j}$. This shows that the outer measure of S_j is $\leq \left(\frac{2}{3}\right)^j + \frac{1}{100} 3^{-j}$. Hence the outer measure of the Cantor set is $\leq \left(\frac{2}{3}\right)^j + \frac{1}{100} 3^{-j}$. Since this is true for every $j > 0$, we conclude that the outer measure of the Cantor set is 0.

2. Let $\varepsilon > 0$. The hypothesis $m^*(E) = 0$ means that there exist open intervals I_j with $E \subseteq \bigcup_j I_j$ and $\sum \ell(I_j) < \varepsilon$. But then

$$m(E) \leq m\left(\bigcup I_j\right) \leq \sum_j m(I_j)$$
$$= \sum_j \ell(I_j) < \varepsilon.$$

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Since this true for every $\varepsilon > 0$, we conclude that $m(E) = 0$.

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3. Let $I = [0,1]$. Let A satisfy $m^*(A) < \infty$.
It suffices to show that

$$m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I).$$

Let $n \in \mathbb{N}$. Set

$$I_n = \{x \in I : \text{dist}(x, {}^cI) > 1/n\}.$$

So $I_n \subseteq I$. Since $I \setminus I_n$ is the union of two intervals each having length $1/n$, we see that

$$m^*(I \setminus I_n) \to 0 \text{ as } n \to \infty.$$

Now $A \supseteq (A \cap I_n) \cup (A \setminus I)$

and $\text{dist}(A \cap I_n, A \setminus I) \geq 1/n$.

So we know from Proposition 6.6 in the text that

$$m^*(A) \geq m^*((A \cap I_n) \cup (A \setminus I))$$
$$= m^*(A \cap I_n) + m^*(A \setminus I).$$

Also $A \cap I = (A \cap I_n) \cup (A \cap (I \setminus I_n))$

The subadditivity and monotonicity of m^* now tells us that

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$$m^*(A \cap I_n) \leq m^*(A \cap I) \leq m^*(A \cap I_n) + m^*(I \setminus I_n)$$

So $m^*(A \cap I) = \lim_{n \to \infty} m^*(A \cap I_n)$.

Thus $m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I)$.

That proves the result.

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4. For $a \in \mathbb{R}$, let $I_a = \{x \in \mathbb{R} : a \le x \le a+1\}$. Then each I_a has Lebesgue measure 1 and $I_a \ne I_{a'}$ when $a \ne a'$.

5. Let F be any measurable set. Then

$$\mu(E \cap F) = \mu(P \cap (E \cap F))$$

which is ≥ 0 because P is positive. Hence E is positive

6. Let $A = \{a_j\}_{j=1}^{\infty}$ be countable. Let $\varepsilon > 0$, let $I_j = (a_j - \varepsilon/2^j, a_j + \varepsilon/2^j)$. Then

$\bigcup_j I_j \supseteq A$. So

$$m(A) \le m\left(\bigcup_j I_j\right) \le \sum_j m(I_j) \le \sum_j \frac{\varepsilon}{2^{j-1}}$$

$$= 2\varepsilon.$$

Since this is true for every $\varepsilon > 0$, we conclude that $m(A) = 0$.

7. Let C be the Cantor ternary set. The complement of C in $[0,1]$ consists of

1 interval of length $1/3$
2 intervals of length $1/9$

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4 intervals of length $1/27$
etc.

So $[0,1] \setminus C$ has measure 1. We conclude that C has measure 0. And it is uncountable.

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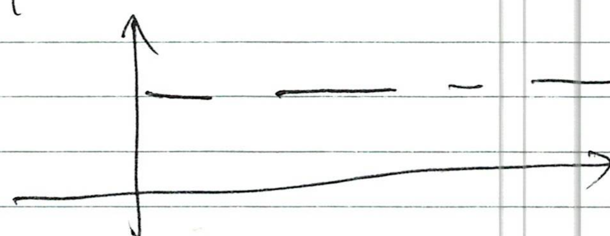
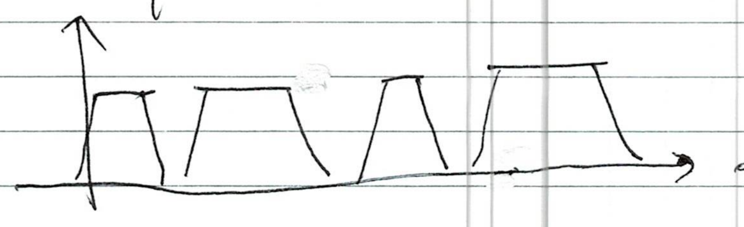
8. We know that we can approximate f in L^1 norm by a single function s. And each characteristic function in s can be approximated by the characteristic function of a Borel set. And each characteristic function of a Borel set can be approximated by a characteristic function of a disjoint union of intervals.



Such a function can in turn be approximated by a continuous function



9. $$\int |Tf(x)|dx = \int\left|\int f(x-t)\varphi(t)\,d\lambda(t)\right|d\lambda(x)$$
$$\le \iint |f(x-t)||\varphi(t)|\,d\lambda(t)\,d\lambda(x)$$
$$= \int |f(x-t)|\,d\lambda(x)\int|\varphi(t)|\,d\lambda(t)$$
$$< \infty.$$

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Let $f_a(x) = f(x-a)$,

Then

$$T(f_a)(x) = \int f_a(x-t)\,\varphi(t)\,dt$$

$$= \int f(x-t-a)\,\varphi(t)\,dt$$

$$= \left[\int f(x-t)\,\varphi(t)\,dt\right]_a$$

$$= (Tf)_a(x).$$

20. Fix $x>0$. If j is large enough then

$\frac{2}{j} < x$ so $g_j(x) = 0$. So $\lim_{j\to\infty} g_j(x) = 0$.

Similar for $x \le 0$.

Now matter how large j is, $g_j\left(\frac{3}{2j}\right) = j$.

So g_j does not converge uniformly to 0.

$$\left(\int |g_j(x) - 0|^2\,dx\right)^{\frac{1}{2}} = \left(\int_{1/j}^{2/j} j^2\,dx\right)^{1/2} = j^{1/2} \not\to 0.$$

So no convergence in L^2.

Let $\alpha > 0$. If $j > \alpha$ then

$\lambda(\{x : |g_j(x) - 0| \ge \alpha\})$

$= \lambda(\{x : |j - 0| \ge \alpha\}) = \frac{1}{j} \to 0$

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So $g_j \to 0$ in measure.

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11. Use counting measure on $\mathbb{N}$. Now just invoke Tonelli's theorem.

12. Now $N \setminus S \subseteq N$. So $N \setminus S$ is measurable and has measure 0. If $N \cup S$ were measurable then

$$(N \cup S) \setminus (N \setminus S) = S$$

would be measurable. And that is not true. So $N \cup S$ is not measurable.

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