1. At the \( j \)th step in the construction of the Cantor set, we have a compact set \( S_j \) consisting of \( 2^j \) pairwise disjoint intervals having total length \( 2^j/3^j \). Thus, it is easy to cover \( S_j \) by \( 2^j \) pairwise disjoint open intervals of length \( (2/3)^j + \frac{1}{100} \cdot 3^{-j} \).

This shows that the outer measure of \( S_j \) is
\[
\leq \left( \frac{2}{3} \right)^j + \frac{1}{100} \cdot 3^{-j}.
\]
Hence, the outer measure of the Cantor set is
\[
\leq \left( \frac{2}{3} \right)^j + \frac{1}{100} \cdot 3^{-j}.
\]
Since this is true for every \( j > 0 \), we conclude that the outer measure of the Cantor set is 0.

2. Let \( \varepsilon > 0 \). The hypothesis \( m^* (E) = 0 \) means that there exist open intervals \( I_j \) with \( E \subseteq \bigcup I_j \) and \( \sum_j l (I_j) < \varepsilon \).

But then
\[
m^*(E) \leq m^*(\bigcup I_j) \leq \sum_j m^*(I_j) \leq \varepsilon \leq l (I_j) < \varepsilon.
\]

Since this is true for every \( \varepsilon > 0 \), we conclude that \( m^*(E) = 0 \).
3. Let \( I = [0, 1] \). Let \( A \) satisfy \( m^*(A) < \infty \). It suffices to show \( m^* \)
\[ m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I). \]

Let \( n \in \mathbb{N} \). Set
\[ I_n = \{ x \in I : \text{dist}(x, \partial I) > 1/n \}. \]
So \( I_n \subset I \). Since \( I \setminus I_n \) is the union of two intervals each having length \( 1/n \), we see that
\[ m^*(I \setminus I_n) \to 0 \quad \text{as} \quad n \to \infty. \]

Now \( A \supseteq (A \cap I_n) \cup (A \setminus I) \)
and \( \text{dist}(A \cap I_n, A \setminus I) \geq 1/n \).
So we know from Proposition 6.6 in the text that
\[ m^*(A) \geq m^*((A \cap I_n) \cup (A \setminus I)) \]
\[ \geq m^*(A \cap I_n) + m^*(A \setminus I). \]

Also \( A \cap I = (A \cap I_n) \cup (A \cap I \setminus I_n) \)
The subadditivity and monotonicity of \( m^* \)
how tells us that
\[ m^*(A \cap I_n) \leq m^*(A \cap I) \leq m^*(A \cap I_n) + m^*(I \setminus I_n). \]
So \( m^*(A \cap I) = \lim_{n \to \infty} m^*(A \cap I_n) \).
Thus \( m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I) \).
That \( \geq \) gives the result.
4. For $a \in \mathbb{R}$ let $I_a = \{x \in \mathbb{R} : a \leq x \leq a + 1\}$. Then each $I_a$ has Lebesgue measure 1 and $I_a \neq I_b$ when $a \neq b$.

5. Let $F$ be any measurable set. Then

$$m(E \cap F) = m(P \cap (E \cap F))$$

which is $\geq 0$ because $P$ is positive. Hence

$E$ is positive.

6. Let $A = \bigcup_{j=1}^{\infty} I_j$ be countable. Let $\varepsilon > 0$, let $E_j = (a_j - \frac{\varepsilon}{2}2^j, a_j + \frac{\varepsilon}{2}2^j)$. Then

$$\bigcup_{j=1}^{\infty} E_j = A.$$ So

$$m(A) \leq \sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2}2^j = 2\varepsilon.$$ Since this is true for every $\varepsilon > 0$, we conclude that $m(A) = 0$.

7. Let $C$ be the Cantor ternary set. The complement of $C$ in $[0,1]$ consists of

- Interval of length $1/3$
- 2 intervals of length $1/9$
- 4 intervals of length $1/27$
- etc.

So $[0,1] \setminus C$ has measure 1. We conclude that $C$ has measure 0, and it is uncountable.
8. We know that we can approximate a function \( L \) by a single function \( \varphi \). And each characteristic function \( \chi_{S} \) can be approximated by its characteristic function of a Borel set. And each characteristic function of a Borel set can be approximated by a characteristic function of a disjoint union of intervals.

\[
\begin{align*}
\varphi(x) &= \sum_{i} \chi_{A_i}(x) \varphi_i(x) \\
&\leq \sum_{i} |f(x-t_i)| |\varphi(t_i)| d\lambda(t_i) d\lambda(x)
\end{align*}
\]

\[
= \sum_{i} f(x-t_i) d\lambda(x_i) \int |\varphi(t)| d\lambda(t)
\]

\[
< \infty.
\]
\[ \text{Let } f_\alpha(x) = f(x - \alpha), \]

Then,

\[ T(f_\alpha)(x) = \int f_\alpha(x-t) \varphi(t) \, dt \]

\[ = \int f(x-t-\alpha) \varphi(t) \, dt \]

\[ = \left[ \int f(x-t) \varphi(t) \, dt \right] \bigg|_{\alpha}^{0} \]

\[ = \left( T-f \right)_\alpha(x). \]

10. Fix \( x \neq 0 \). If \( j \) is large enough,

\[ \frac{2}{j} < x \quad \text{so} \quad g_j(x) = 0, \quad \text{so} \lim_{j \to \infty} g_j(x) = 0. \]

Similarly for \( x < 0 \),

Now matter how large \( j \) is, \( g_j(x) = \frac{2}{j} \).

So \( g_j \) does not converge uniformly to \( 0 \).

\[ \int |g_j(x)| \, dx \leq \frac{2}{j} x^2 \int \frac{1}{j^2} \, dx = \frac{1}{j} \cdot \frac{1}{2} x^2 \to 0. \]

So no convergence in \( L^2 \).

Let \( \alpha > 0 \), \( j > x \) large,

\[ \lambda \left( \{ x : |g_j(x) - 0| \geq \alpha \} \right) \]

\[ = \lambda \left( \{ x : |j-o| \geq \alpha \} \right) \to 0 \quad \text{as} \quad \alpha \to 0 \]

So \( g_j \to 0 \) in measure.
11. Use counting measure on N. Now just invoke Tonelli's Theorem.

12. Now N \cap S = \emptyset. So N \cup S is measurable and its measure 0.

If N \cup S were measurable then

\[(N \cup S) \setminus (N \cap S) = S\]

would be measurable. And that is not true, so N \cup S is not measurable.