Math 416
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Figure: This is your instructor.

Complex Analysis

In this class we will frequently refer to a *domain* or a *region* $U \subseteq \mathbb{C}$. Usually this will mean that U is an open set and that U is connected.

Holomorphic functions are a generalization of complex polynomials. But they are more flexible objects than polynomials. The collection of all polynomials is closed under addition and multiplication. However, the collection of all holomorphic functions is closed under reciprocals, inverses, exponentiation, logarithms, square roots, and many other operations as well.

There are several different ways to introduce the concept of holomorphic function. They can be defined by way of power series, or using the complex derivative, or using partial differential equations. We shall touch on all these approaches; but our initial definition will be by way of partial differential equations.

If $U \subseteq \mathbb{R}^2$ is a region and $f : U \to \mathbb{R}$ is a continuous function, then f is called C^1 (or *continuously differentiable*) on U if $\partial f/\partial x$ and $\partial f/\partial y$ exist and are *continuous* on U. We write $f \in C^1(U)$ for short.

More generally, if $k \in \{0, 1, 2, ...\}$, then a real-valued function f on U is called C^k (k times continuously differentiable) if all partial derivatives of f up to and including order k exist and are continuous on U. We write in this case $f \in C^k(U)$. In particular, a C^0 function is just a continuous function.

Example: Let $D \subseteq \mathbb{C}$ be the unit disc, $D = \{z \in \mathbb{C} : |z| < 1\}$. The function $\varphi(z) = |z|^2 = x^2 + y^2$ is C^k for every k. This is so just because we may differentiate φ as many times as we please, and the result is continuous. In this circumstance we sometimes write $\varphi \in C^\infty$.

By contrast, the function $\psi(z) = |z|$ is not even C^1. For the restriction of ψ to the real axis is $\widetilde{\psi}(x) = |x|$, and this function is well known not to be differentiable.

A function $f = u + iv : U \to \mathbb{C}$ is called C^k if both u and v are C^k.

If f is *any* complex-valued function, then we may write $f = u + iv$, where u and v are real-valued functions.

Example: Consider

$$f(z) = z^2 = (x^2 - y^2) + i(2xy); \qquad (1)$$

in this example $u = x^2 - y^2$ and $v = 2xy$. We refer to u as the *real part* of f and denote it by $\mathrm{Re}\, f$; we refer to v as the *imaginary part* of f and denote it by $\mathrm{Im}\, f$. □

Now we formulate the notion of “holomorphic function” in terms of the real and imaginary parts of f :

Let $U \subseteq \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ a C^1 function. Write

$$f(z) = u(x, y) + iv(x, y), \tag{2}$$

with u and v real-valued functions. Of course $z = x + iy$ as usual. If u and v satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

at every point of U, then the function f is said to be *holomorphic*. The first order, linear partial differential equations in (21) are called the *Cauchy–Riemann equations*. A practical method for checking whether a given function is holomorphic is to check whether it satisfies the Cauchy–Riemann equations. Another practical method is to check that the function can be expressed in terms of z alone, with no $\overline{z}$'s present.

Example: Let $f(z) = z^2 - z$. Then we may write

$$f(z) = (x+iy)^2-(x+iy) = (x^2-y^2-x)+i(2xy-y) \equiv u(x,y)+iv(x,y)\,.$$

Then we may check directly that

$$\frac{\partial u}{\partial x} = 2x - 1 = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}\,.$$

We see, then, that f satisfies the Cauchy–Riemann equations so it is holomorphic. Also observe that f may be expressed in terms of z alone, with no $\overline{z}$s.

Example: Define

$$\begin{aligned} g(z) &= |z|^2 - 4z + 2\overline{z} \\ &= z \cdot \overline{z} - 4z + 2\overline{z} \\ &= (x+iy)\cdot(x-iy) - 4\cdot(x+iy) + 2(x-iy) \\ &= (x^2+y^2-2x) + i(-6y) \\ &\equiv u(x,y) + iv(x,y) \, . \end{aligned}$$

Then

$$\frac{\partial u}{\partial x} = 2x - 2 \neq -6 = \frac{\partial v}{\partial y} \, .$$

Also

$$\frac{\partial u}{\partial y} = 2y \neq 0 = -\frac{\partial v}{\partial x} .$$

We see that *both* Cauchy–Riemann equations fail. So g is not holomorphic. We may also observe that g is expressed both in terms of z and $\bar{z}$—another sure indicator that this function is not holomorphic. □

Derivatives

We define, for $f = u + iv : U \to \mathbb{C}$ a C^1 function,

$$\begin{aligned} \frac{\partial}{\partial z} f &\equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \overline{z}} f &\equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f \\
&\equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) .
\end{aligned}
$$

If $z = x + iy$, $\overline{z} = x - iy$, then one can check directly that

$$
\frac{\partial}{\partial z} z = 1 , \qquad \frac{\partial}{\partial z} \overline{z} = 0 , \tag{4}
$$

$$
\frac{\partial}{\partial \overline{z}} z = 0 , \qquad \frac{\partial}{\partial \overline{z}} \overline{z} = 1. \tag{5}
$$

If a C^1 function f satisfies $\partial f/\partial z \equiv 0$ on an open set U, then f does not depend on z (but it *can* depend on $\bar{z}$). If instead f satisfies $\partial f/\partial \bar{z} \equiv 0$ on an open set U, then f does not depend on $\bar{z}$ (but it *does* depend on z). The condition $\partial f/\partial \bar{z} = 0$ is just a reformulation of the Cauchy–Riemann equations. We work out the details of this claim below. Now we look at some examples to illustrate the new ideas.

Example: Review our earlier example. Now let us examine that same function using our new criterion with the operator $\partial/\partial\overline{z}$. We have

$$\frac{\partial}{\partial\overline{z}}f(z) = \frac{\partial}{\partial\overline{z}}\left(z^2 - z\right) = 2z\frac{\partial z}{\partial\overline{z}} - \frac{\partial z}{\partial\overline{z}} = 0 - 0 = 0\,.$$

We conclude that f is holomorphic.

Example: Review another earlier example. Now let us examine that same function using our new criterion with the operator $\partial/\partial\overline{z}$. We have

$$\frac{\partial}{\partial\overline{z}}g(z) = \frac{\partial}{\partial\overline{z}}\left(|z|^2 - 4z + 2\overline{z}\right) = \frac{\partial}{\partial\overline{z}}\left(z \cdot \overline{z} - 4z + 2\overline{z}\right) = z + 2 \neq 0\,.$$

We conclude that g is *not* holomorphic. □

Definition of Holomorphic Function

Functions f that satisfy $(\partial/\partial\overline{z})f \equiv 0$ are the main concern of complex analysis. A continuously differentiable (C^1) function $f : U \to \mathbb{C}$ defined on an open subset U of $\mathbb{C}$ is said to be *holomorphic* if

$$\frac{\partial f}{\partial \overline{z}} = 0 \qquad (6)$$

at every point of U. Note that this last equation is just a reformulation of the Cauchy–Riemann equations. To see this, we calculate:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \overline{z}} f(z) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)[u(z) + iv(z)] \\
&= \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \ i\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right].
\end{aligned}
\tag{25}
$$

Of course the far right-hand side cannot be identically zero unless each of its real and imaginary parts is identically zero. It follows that

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{7}$$

and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0\,. \tag{8}$$

These are the Cauchy–Riemann equations.

Example: The function $h(z) = z^3 - 4z^2 + z$ is holomorphic because

$$\frac{\partial}{\partial \overline{z}} h(z) = 3z^2 \frac{\partial z}{\partial \overline{z}} - 4 \cdot 2z \frac{\partial z}{\partial \overline{z}} + \frac{\partial z}{\partial \overline{z}} = 0 \ .$$