Math 416
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Figure: This is your instructor.

Complex Analysis

Examples of Holomorphic Functions

Certainly any polynomial in z (*without* $\overline{z}$) is holomorphic. And the reciprocal of any polynomial is holomorphic, as long as we restrict attention to a region where the polynomial does not vanish.

Earlier in this book we have discussed the complex function

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} .$$

One may calculate directly, just differentiating the power series term-by-term, that

$$\frac{\partial}{\partial z} e^z = e^z .$$

In addition,

$$\frac{\partial}{\partial \overline{z}} e^z = 0 ,$$

so the exponential function is holomorphic.

The Complex Derivative

Let $U \subseteq \mathbb{C}$ be open, $P \in U$, and $g : U \setminus \{P\} \to \mathbb{C}$ a function. We say that

$$\lim_{z \to P} g(z) = \ell \, , \quad \ell \in \mathbb{C} \, , \tag{1}$$

if, for any $\epsilon > 0$ there is a $\delta > 0$ such that when $z \in U$ and $0 < |z - P| < \delta$ then $|g(z) - \ell| < \epsilon$. Notice that, in this definition of limit, the point z may approach P in an arbitrary manner—from any direction. See the figure below. Of course the function g is *continuous* at $P \in U$ if $\lim_{z \to P} g(z) = g(P)$.

We say that f possesses the *complex derivative* at P if

$$\lim_{z \to P} \frac{f(z) - f(P)}{z - P} \tag{2}$$

exists. In that case we denote the limit by $f'(P)$ or sometimes by

$$\frac{df}{dz}(P) \qquad \text{or} \qquad \frac{\partial f}{\partial z}(P). \tag{3}$$

This notation is consistent with that introduced earlier: for a *holomorphic function*, the complex derivative calculated according to our earlier formula or according to our current formula is just the same. We shall say more about the complex derivative in what follows.

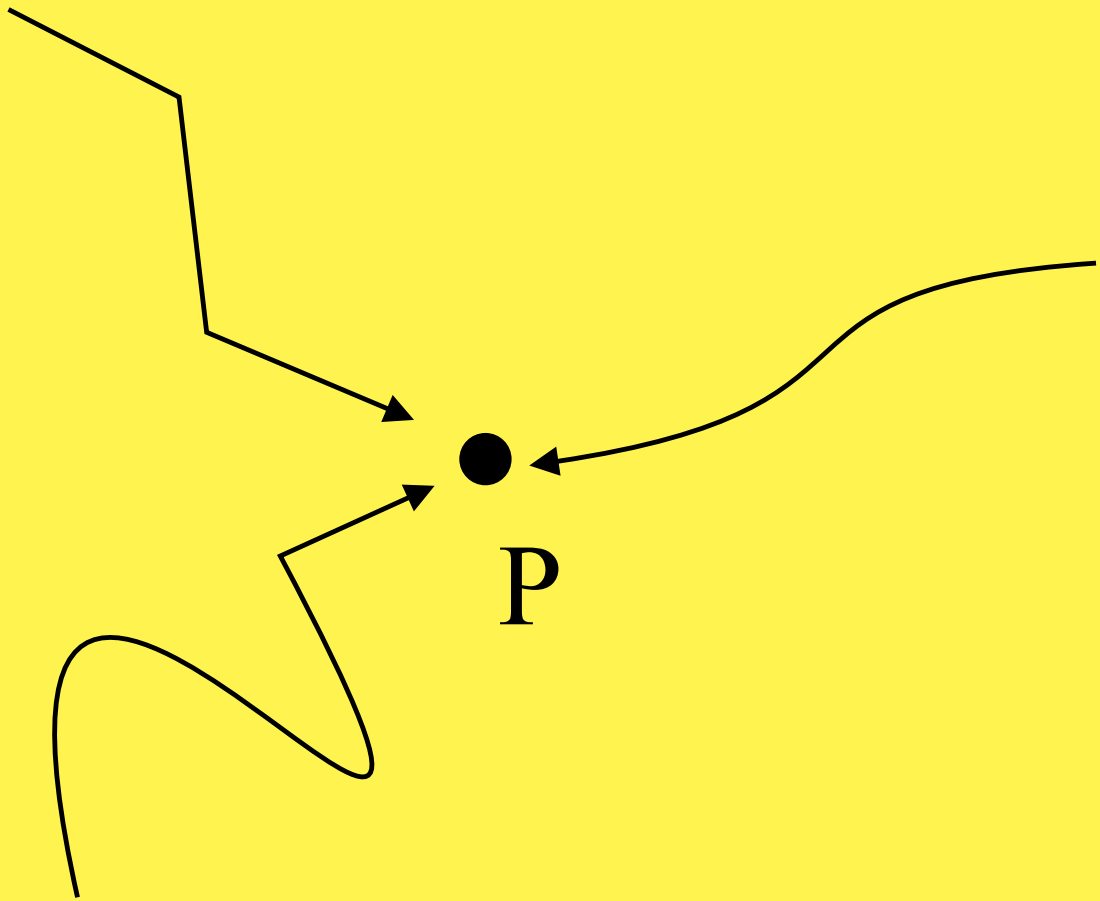


Figure: The point may approach P arbitrarily.

We repeat that, in calculating the limit in (1), z must be allowed to approach P from *any* direction (refer to the figure). As an example, the function $g(x, y) = x - iy$—equivalently, $g(z) = \bar{z}$—does *not* possess the complex derivative at 0. To see this, calculate the limit

$$\lim_{z \to P} \frac{g(z) - g(P)}{z - P} \tag{4}$$

with z approaching $P = 0$ through values $z = x + i0$. The answer is

$$\lim_{x \to 0} \frac{x - 0}{x - 0} = 1. \tag{5}$$

If instead z is allowed to approach $P = 0$ through values $z = iy$, then the value is

$$\lim_{z \to P} \frac{g(z) - g(P)}{z - P} = \lim_{y \to 0} \frac{-iy - 0}{iy - 0} = -1. \tag{6}$$

Observe that the two answers do not agree. *In order for the complex derivative to exist, the limit must exist and assume only one value no matter how z approaches P.* Therefore, this example g does not possess the complex derivative at $P = 0$. In fact a similar calculation shows that this function g does *not* possess the complex derivative at any point.

If a function f possesses the complex derivative at every point of its open domain U, then f is holomorphic. This definition is equivalent to definitions given earlier. In fact, from an historical perspective, it is important to recall a theorem of Goursat (see the Appendix in [GRK]). It says that if a function f possesses the complex derivative at each point of an open region $U \subseteq \mathbb{C}$ then f is in fact continuously differentiable on U. One may then verify the Cauchy–Riemann equations, and it follows that f is holomorphic by any of our definitions thus far.

Alternative Terminology for Holomorphic Functions

Some books use the word "analytic" instead of "holomorphic." Still others say "differentiable" or "complex differentiable" instead of "holomorphic." The use of the term "analytic" derives from the fact that a holomorphic function has a local power series expansion about each point of its domain. In fact this power series property is a complete characterization of holomorphic functions; we shall discuss it in detail below. The use of "differentiable" derives from properties related to the complex derivative. These pieces of terminology and their significance will all be sorted out as the book develops. Somewhat archaic terms for holomorphic functions, which may be found in older texts, are "regular" and "monogenic."

Another piece of terminology that is applied to holomorphic functions is "conformal" or "conformal mapping." "Conformality" is an important geometric property of holomorphic functions that make these functions useful for modeling incompressible fluid flow and other physical phenomena. We shall discuss conformality later in the course. We shall treat physical applications of conformality in subsequent lectures.

The Relationship of Holomorphic and Harmonic Functions

A C^2 function u is said to be *harmonic* if it satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0. \tag{7}$$

This equation is called *Laplace's equation*, and is frequently abbreviated as

$$\triangle u = 0. \tag{8}$$

Example: The function $u(x,y) = x^2 - y^2$ is harmonic. This assertion may be verified directly:

$$u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \left(\frac{\partial^2}{\partial x^2}\right) x^2 - \left(\frac{\partial^2}{\partial y^2}\right) y^2 = 2 - 2 = 0\,.$$

A similar calculation shows that $v(x,y) = 2xy$ is harmonic. For

$$v = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) 2xy = 0 + 0 = 0\,. \qquad \square$$

Example: The function $\widetilde{u}(x,y) = x^3$ is *not* harmonic. For

$$\widetilde{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) x^3 = 6x \neq 0\,.$$

Likewise, the function $\widetilde{v}(x,y) = \sin x - \cos y$ is *not* harmonic. For

$$\begin{aligned}
\widetilde{v} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \widetilde{v} \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) [\sin x - \cos y] \\
&= -\sin x + \cos y \neq 0\,.
\end{aligned}$$

Holomorphic and Harmonic Functions

If f is a holomorphic function and $f = u + iv$ is the expression of f in terms of its real and imaginary parts, then both u and v are harmonic. The easiest way to see this is to begin with the equation

$$\frac{\partial}{\partial \bar{z}} f = 0 \tag{9}$$

and to apply $\partial/\partial z$ to both sides.

The result is

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f = 0 \tag{10}$$

or

$$\left(\frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]\right)\left(\frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]\right)[u + iv] = 0\,. \tag{11}$$

Multiplying through by 4, and then multiplying out the derivatives, we find that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[u + iv] = 0\,. \tag{12}$$

We may now distribute the differentiation and write this as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u \quad + \quad i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v = 0\,. \tag{13}$$

The only way that the left-hand side can be zero is if its real part is zero and its imaginary part is zero. We conclude then that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \tag{14}$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0\,. \tag{15}$$

Thus u and v are each harmonic.

Example: Let $f(z) = (z + z^2)^2$. Then f is certainly holomorphic because it is defined using only zs, and no $\bar{z}$s. Notice that

$$\begin{aligned} f(z) &= z^4 + 2z^3 + z^2 \\ &= [x^4 - 6x^2y^2 + y^4 + 2x^3 - 6xy^2 + x^2 - y^2] \\ &\quad + i[-4xy^3 + 4x^3y + 6x^2y - 2y^3 + 2xy] \\ &\equiv u + iv\,. \end{aligned}$$

We may check directly that

$$\triangle u = 0 \qquad \text{and} \qquad \triangle v = 0\,.$$

Hence the real and imaginary parts of f are each harmonic.