# HOMEWORK 7 SOLUTIONS

pp. 162-164

1. If $f$ had infinitely many zeros in $D(P,r)$ then the zeros would have an accumulation point $z_0$ in the interior of the domain of $f$. So $f$ would be $\equiv 0$, Contradiction.

2. $$\frac{1}{2\pi i}\oint_{\gamma} \frac{f'(z)}{f(z)} \cdot g(z)\, dz$$

Say $f$ has a zero of order $n_j$ at $P_j$. Choose $r_j > 0$ small, $j = 1, \ldots, k$.

Then

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{f'(z)}{f(z)} g(z)\, dz = \sum_{j=1}^{k} \frac{1}{2\pi i}\oint_{\partial D(P_j, r_j)} \frac{n_j}{z - P_j} g(z)\, dz$$

$$= \sum_{j=1}^{n} n_j\, g(P_j)$$

by the Cauchy integral formula.

3. $$\frac{(e^z)'}{e^z} = \frac{e^z}{e^z} = 1.$$ Hence

$$\frac{1}{2\pi i}\oint_{\partial D(0,R)} \frac{(e^z)'}{e^z}\, dz = \frac{1}{2\pi i}\oint_{\partial D(0,R)} 1\, dz = 0$$

by the Cauchy integral theorem. So $e^z$ has no zeros in $D(0, R)$, any $R>0$.

So $e^z$ has no zeros.

5. $$\frac{\partial}{\partial z} g(z) = \frac{f'(z)}{f(z)}$$

Define $$g(z) = \oint_{\gamma_z} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta .$$

where $\gamma_z$ is any smooth curve in $D(0, 1)$ connecting $0$ to $z$. It is easy to see, using the Cauchy integral theorem, that this defn. is independent of the choice of $\gamma_z$. It is also easy to check, using the Cauchy integral theorem, that

$$\frac{\partial}{\partial z} g(z) = \frac{f'(z)}{f(z)}.$$

3

pp. 167 – 168

1. $f(z) = z^3 + z/2$. Let $g(z) = z^3$. Then

$|f(z) - g(z)| = |z/2| < |f(z)| + |g(z)|$ on $\partial D(0, 1)$

So $f$ has the same number of zeros as $g$

in $D(0, 1)$. That number is 3.

2. $f_j(z) = e^{z/j}$

Each of these fns. is non vanishing.

The limit function is $g(z) = e^0 \equiv 1$.

Also nonvanishing.

That is consistent with Hurwitz.

4. $f_j(z) = \cos(z/j)$

On the unit disc, if $j$ is large, then $f_j$ is non vanishing. The limit fn. is

$g(z) = \cos 0 \equiv 1$. Also nonvanishing.

This is consistent with Hurwitz.

5. $f(z) = e^z$. Let $g(z) = 1$. Then

$$|f(z) - g(z)| = \left|\sum_{j=1}^{\infty} \frac{z^j}{j!}\right| = |z|\left|\sum_{j=1}^{\infty} \frac{z^{j-1}}{j!}\right|$$

$$= \left|\sum_{j=0}^{\infty} \frac{z^j}{(j+1)!}\right| = \left|\sum_{j=1}^{\infty} \frac{z^j}{j!} + \sum_{j=0}^{\infty} z^j\left(\frac{1}{(j+1)!} - \frac{1}{j!}\right)\right|$$

$$\leq |e^z| + 1 = |f(z)| + |g(z)|.$$

So $f, g$ have same no. zeros in $D(0,1)$ – namely none.

pp. 171-172

1. We see that, if $g$ does not vanish on $\partial U$, then $\left|\frac{f}{g}\right| \leq 1$ on $\partial U$. So maxim principle $\Longrightarrow$ $\left|\frac{f}{g}\right| \leq 1$ on $U$. So $|f| \leq |g|$ on $U$.

2. If $f$ took a point of $\partial D$ to an interior point of $D$, then $|f^{-1}|$ would have a local interior maxim, which is impossible. So

$$f^{-1}(\partial D) \subseteq \partial D.$$

Similarly, $f(\partial D) \subseteq \partial D$.

3. $f(z) = z \cdot (z - \frac{1}{2}) \cdot (z - \frac{1}{3}i)$.

Minima for $|f|$ are at $0, \frac{1}{2}, \frac{1}{3}$.

5. If $f(z) = f(w)$ then

a) $$i\frac{1-z}{1+z} = i\frac{1-w}{1+w}$$

$$\frac{1-z}{1+z} = \frac{1-w}{1+w}$$

$$(1-z)(1+w) = (1+z)(1-w)$$

$$1 - z + w - zw = 1 + z - w - zw$$

$$-z + w = z - w$$

$$2w = 2z$$

$$w = z.$$

b) $$f(e^{i\theta}) = i\frac{1-e^{i\theta}}{1+e^{i\theta}} = i\frac{(1-e^{i\theta})(1+e^{-i\theta})}{(1+e^{i\theta})(1+e^{-i\theta})}$$

$$= \frac{i(1 - 2i\sin\theta - 1)}{2 + 2\cos\theta} = \frac{i(-i\sin\theta)}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$\in \partial \overline{U}$.

c) $f(0) = i\cdot\frac{1-0}{1+0} = i$

d) If $w \in U$, set

$$i\cdot\frac{1-z}{1+z} = w$$

$$i(1-z) = w + wz$$

$$i - w = z(w+i)$$

$$\frac{i-w}{i+w} = z$$

It is straightforward to check that if $w \in U$ then

$$\frac{i-w}{i+w} \in D.$$

e) Because of b) and the maximum principle,

$f(D(0,1)) \subseteq U$.