Math 416
Krantz

Spring, 2022
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SECOND MIDTERM

General Instructions: Read the statement of each problem carefully. On each problem you should show your work. If you only write the answer then you will not receive full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

(10 points) **1.** Suppose that f is an entire function that satisfies the estimate

$$|f(z)| \le C \cdot (1 + |z|^k)$$

for all z and for some positive integer k. Prove that f must be a polynomial of degree at most k. [**Hint:** Use the Cauchy estimates.]

Let $P \in \mathbb{C}$ and $R > |P|$. Then $D(P, R) \subseteq D(0, 2R)$.

By Cauchy estimates,

$$|f^{(k+1)}(P)| \le \frac{C(1 + (2R)^k) \cdot (k+1)!}{R^{k+1}}$$

Letting $R \to \infty$ gives $f^{(k+1)}(P) = 0 \quad \forall P$.

Hence f is a polynomial of degree at most k.

(10 points) **2.** Suppose that f is a holomorphic function on the unit disc and that

$$\mathrm{Re} f(0) \geq \mathrm{Re} f(z)$$

for all z in the unit disc. Prove that f must be constant. [**Hint:** Consider an exponential function.]

Let $F(z) = e^{f(z)}$. Then

$|F(0)| = e^{\mathrm{Re}\, f(0)} \geq e^{\mathrm{Re}\, f(z)} = |F(z)|$ for all $z \in D$.

By the maximum principle, F is constant.

Hence f is constant.

(10 points) **3.** What is the residue of the function

$$f(z) = \frac{z}{\sin^2 z}$$

at the origin?

$\mathrm{Res}_f(0) = z \cdot \frac{z}{\sin^2 z}\Big|_{z=0} = 1.$

(10 points) **4.** Show that the function

$$g(z) = \frac{1-\cos^2 z}{z^2}$$

has a removable singularity at the origin.

$$g(z) = \frac{\sin^2 z}{z^2} = \frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots\right)^2}{z^2}$$

$$= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots\right)^2$$

which is holomorphic in a neighborhood of the origin.

(10 points) **5.** Use the calculus of residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Let $f(z) = \frac{1}{1+z^2}$.

For $R > 1$, let

$$\gamma_1^R(t) = t, \quad -R \le t \le R$$
$$\gamma_2^R(t) = Re^{it}, \quad 0 \le t \le \pi.$$

Then let γ be the union of γ_1^R and γ_2^R.

We have

$$2\pi i \, \mathrm{Res}_f(i) = \left.\frac{z-i}{1+z^2}\right|_{z=i} = 2\pi i \frac{1}{2i} = \pi$$

$$\| $$

$$\oint_\gamma f(z)\,dz = \int_{\gamma_1^R} f(z)\,dz + \int_{\gamma_2^R} f(z)\,dz = A + B.$$

Now $B \to 0$ and $\int_{\gamma_1^R} f(z)\,dz \to \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$

(10 points) **6.** Let each f_j be entire and non-vanishing. Suppose that $f_j \to f$ uniformly on compact sets and that $f(0) = 1$. Prove that in fact f vanishes nowhere.

By Hurwitz's Theorem, f is either identically 0 or nonvanishing. Since $f(0) = 1$, the second instance applies.

(10 points) **7.** What is a Möbius transformation? What is the inverse of that Möbius transformation?

Let $a \in \mathbb{C}$, $|a| < 1$. The corresponding Möbius transformation is

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a} z},$$

The inverse of φ_a is

$$\varphi_{-a}(z) = \frac{z + a}{1 + \bar{a} z}.$$

(10 points) **8.** What kind of singularity does the function $e^{1/z}$ have at infinity?

$G(z) = f(\frac{1}{z}) = e^{z}$ has a removable singularity at 0. So $f(z) = e^{1/z}$ has a removable singularity at ∞.

(10 points) **9.** Give the first three terms of the Laurent expansion of

$$f(z) = \frac{z \cos z}{\sin^2 z}$$

about the origin.

$$\frac{z\cos z}{\sin^2 z} = \frac{z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\right)}{z^2\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots\right)^2}$$

$$= \frac{\left(z - \frac{z^3}{2!} + \frac{z^5}{4!} - + \cdots\right)}{z^2} \cdot \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots\right)^{-2}$$

$$= \left(\frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - + \cdots\right)$$

(10 points) **10.** Prove that if f is holomorphic on the unit disc and

$$\frac{\partial^j}{\partial z^j} f(0) = 0$$

for all $j = 0, 1, 2, \ldots$, then f is identically equal to 0 on the unit disc.

The power series expansion of f about 0 is identically 0. This power series converges uniformly to f on every disc $\overline{D}(0,r)$, $0<r<1$. So $f \equiv 0$.