1. Consider $K_n$, $n \geq 3$. There are $\binom{n}{k}$ ways of choosing $k$ vertices from $n$. There are $k!$ ways of ordering the vertices to form a cycle.

We want distinct cycles, and each cycle occurs have $k$ times (i.e., the same order but beginning at a different vertex). So we have

$$\frac{1}{k} \binom{n}{k} \cdot k! = \binom{n}{k} \cdot (k-1)!$$

cycles on $k$ vertices. The total number of cycles is

$$\sum_{k=3}^{n} \binom{n}{k} \cdot (k-1)!$$

2. Let $G$ be a given graph. Let $G_1$, $G_2$ be even subgraphs. Let $v \in V(G)$. Let $S_1$ be the set of incident edges to $v$ in $G_1$ and $S_2$ be the set of incident edges to $v$ in $G_2$.

Then $|S_1 \Delta S_2| = |S_1 \cap (S_1 \cap S_2)| + |S_2 \cap (S_1 \cap S_2)|$.

Since $G_1$, $G_2$ are even subgraphs, $|S_1|$ and $|S_2|$
are both even. So

\[ \left| S_1 \triangle S_2 \right| \] and \[ \left| S_2 \setminus (S_1 \triangle S_2) \right| \]

have the same parity. Hence \[ \left| S_1 \triangle S_2 \right| \] is even.

Since this assertion holds for every vertex \( v \), \( G_1 \triangle G_2 \) is even.

In case \( G_2 \), \( G_1 \) are both odd, similar reasoning shows that \( \left| S_1 \triangle (S_1 \setminus S_2) \right| \) and \( \left| S_2 \setminus (S_1 \triangle S_2) \right| \) have the same parity.
So \[ \left| S_1 \triangle S_2 \right| \] is even and \( G_1 \triangle G_2 \) is even.

3. There are many different ways to answer this question:

a) \( K_{3,3} \) has six vertices and \( K_{3,2} + K_3 \) has eight vertices. So \( K_{3,3} \) has characteristic polynomial of degree 6 while \( K_{3,2} + K_3 \) has characteristic polynomial of degree 8. The characteristic polynomials are unequal hence the graphs are not isomorphic.
b) The adjacency matrix for $K_{3,3}$ has eigenvalues $3, -3$.

The adjacency matrix for $K_{3,3} + K_3$ has eigenvalues $3, \sqrt{6}, -\sqrt{6}$.

Since the eigenvalues are different, the graphs are not isomorphic.

4. The degree of the characteristic polynomial tells us the number of vertices, so $n(G) = 8$. The coefficient of $x^{n(G) - 2} = x^6$ equals $-e(G)$, so $e(G) = 24$.

Cormen 8.6.6 tells us that the coefficient of $x^{n(G) - 3} = x^5$ is $-2$ times the number of triangles in $G$. So $G$ has 32 triangles.

Taking all this information into account, we must delete 41 edges from $K_8$ to achieve 24 edges and 32 triangles. If we do that in a generic way we get the graph $K_{3,3,2,2}$.

A calculation verifies that this has the right char. poly.
5. The probability that $i$ is fixed, $1 \leq i \leq n$, is

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

Let $X_i$ be the indicator variable of $i$ being fixed. Then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

6. For a vertex to have degree $k$, it must have $k$ neighbors. The probability of this happening is

$$\binom{n-1}{k} p^k (1-p)^{n-1-k}.$$ 

Let $X_i$ be the indicator variable of the $i$th vertex being of degree $k$. Then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \binom{n-1}{k} p^k (1-p)^{k} = n \binom{n-1}{k} p^k (1-p)^{k}.$$
7. Of course there are \((6)\) triples of vertices. Number these triples \(t_i\), \(1 \leq i \leq (6)\), let \(X_i\) be the indicator variable that the triple \(t_i\) is monochromatic. Then

\[
E(\sum X_i) = \sum_i E(X_i) = \sum_i \left[ 1 \cdot p(X_i = 1) + 0 \cdot p(X_i = 0) \right]
\]

\[
= (6) \cdot \frac{1}{3} \cdot 1
\]

\[
= (6) \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}
\]

\[
= \frac{6 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{1}{9} = 5.
\]

8. Let \(A \subseteq V(G)\) be a randomly chosen set. So for each vertex \(v \in G\), \(v \notin A\) with probability \(p\) and \(v \in A\) with probability \(1-p\). Say that the subgraph \(S\) has \(M\) vertices and \(N\) edges. Certainly \(S\) has an independent set of size \(M - N\). Also \(G\) has \(\frac{nd}{2}\) edges. So

\[
E(M - N) = E(M) - E(N) = pn - p^2 \cdot \frac{nd}{2}
\]

\[
= -\frac{nd}{2} \left(p - \frac{1}{2}\right)^2 + \frac{n}{2d}.
\]
This last expression is maximized (using calculus) when \( p = \frac{1}{d} \). Taking \( p = \frac{1}{d} \), we see that

\[
E(M - N) = \frac{n}{2d}.
\]

So there is an independent set with at least \( \frac{n}{2d} \) vertices.

9.2 Every tree is bipartite. We induct on the number \( n(T) \) of vertices.

**Basis Step:** \( n = 1 \). A single vertex is trivially bipartite.

**Inductive Step:** \( n \geq 1 \). Suppose every tree on \( n \) vertices is bipartite, let \( T \) be a tree with \( n + 1 \) vertices. Since \( T \) has at least 2 vertices, \( T \) has at least two leaves, let \( v \) be one of the leaves, let \( T' = T \setminus v \). Then \( T' \) is still a tree, so the inductive hypothesis implies that \( T' \) is bipartite. For \( T \), we delete all vertices from \( T' \) in their partition set, and we assign \( v \) into the partition set that does not contain its only neighbor. This gives a bipartition for \( T \).
b) If course $K_{2,3}$ is bipartite, but it is not a tree because it contains cycles.

10. Let $T$ be a tree with degree $d$. Certainly

$$d = \sum_{v \in V(T)} d(v) = \frac{2e(T)}{n(T)} = \frac{2(n(T)-1)}{n(T)}.$$

Solving for $n(T)$, we see that $n(T) = \frac{2}{2-d}$.

11. Let $G$ be a graph with $n$ vertices, $m$ edges, and $k$ components. If we choose $2$ spanning trees from each component, that uses $m-k$ edges. Since adding an edge to a tree forms just one cycle, each of the remaining $m-n+k$ edges completes a cycle with edges in the spanning forest. The cycles formed are distinct because the edges being added are distinct. So $G$ has $m-n+k$ cycles. Since $k \geq 1$, $G$ has at least $m-n+k$ cycles.
12. The number of vertices in a graph is the sum of the number of vertices in each component. Same for edges. So a graph with fewer edges than vertices must have a component with fewer edges than vertices. Such a component $G^*$ is connected, and has $n(\overline{G^*})-1$ edges, so it is a tree.

13. An edge is a cut-edge iff it belongs to no cycle. So every edge in $G$ is a cut-edge iff $G$ has no cycle. If $G$ is connected, then $G$ is a tree.

Conversely, if $G$ is a tree, then we know it is connected and every edge is a cut-edge.

24. $k = 3$

   \[
   k = 4
   \]

   $k = 5$

   $k = 6$ Impossible.
15. Let \( V_1, V_2, \ldots, V_{n-1} \) be the vertices of \( P_{n-1} \) in order. Let \( x \) be the added vertex that is adjacent to each \( V_j \). Then the spanning tree has diameter \( k \).