Second Midterm

General Instructions: Read each problem carefully. Do only what is requested—nothing more nor less. Always show your work on each problem—unless the problem explicitly tells you not to show work. You will not get full credit for just writing down the answer. The point value of each problem is shown. Use the backs of the pages if you need more space to solve a problem.

The total time on this exam is two hours.

The total points on this exam is 100.

Ask questions if anything is unclear.

There are several TRUE/FALSE and multiple choice questions. For those problems you need not show any work.

No calculators are allowed during this exam. Be sure to put your name and student ID number on the exam.

1. (10 points) Find the center of curvature, the radius of curvature, and the osculating plane for the curve \( r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + tk \) at the point \((-1, 0, \pi)\).

\[
\begin{align*}
\mathbf{r}'(t) &= -\sin t \mathbf{i} + \cos t \mathbf{j} + k, \\
\mathbf{r}''(t) &= -\cos t \mathbf{i} - \sin t \mathbf{j}, \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-\sin t & \cos t & 1 \\
0 & 1 & 0
\end{vmatrix} = \sin t \mathbf{i} - \cos t \mathbf{j} + k,
\end{align*}
\]

Then:

\[
\begin{align*}
\|\mathbf{r}' \times \mathbf{r}''\| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}, \\
\|\mathbf{r}'\| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}, \\
\mathbf{T} &= \frac{-\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}}{\sqrt{2}}, \\
\mathbf{T}' &= \frac{\mathbf{r}''}{\|\mathbf{r}'\|} = \frac{\sin t \mathbf{i} - \cos t \mathbf{j}}{\sqrt{2}}, \\
\mathbf{N} &= \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\sin t \mathbf{i} - \cos t \mathbf{j}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} \sin t \mathbf{i} - \frac{\sqrt{2}}{2} \cos t \mathbf{j}.
\end{align*}
\]
The designated point corresponds to \( t = \pi \).

Then \( N(\pi) = 1 \hat{i} - 0 \hat{j} = \hat{i} \)

Thus the center of curvature is \( \langle -1, 0, \pi \rangle + 2 \hat{i} = \langle 1, 0, \pi \rangle \).

\[
B = N \times T = \frac{1}{2} \times \left( -\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{k} \right) = \det \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 \hat{i} - \frac{1}{\sqrt{2}} \hat{j} - \frac{1}{\sqrt{2}} \hat{k}
\]

The osculating plane is

\[
\left( -\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{k} \right) \cdot \hat{p} \hat{S} = 0
\]

where \( \hat{p} = (-1, 0, \pi) \) and \( \hat{S} = (x, y, t) \).

2. (8 points) The normal component of acceleration for the motion \( \mathbf{r}(t) = t \hat{i} - t^3 \hat{j} + t^2 \hat{k} \)

at the point \((1, -1, 1)\) is given by

(a) \( \sqrt{35}/5 \)

(b) \( \sqrt{36}/6 \)

(c) \( \sqrt{37}/8 \)

(d) \( \sqrt{38}/7 \)

\[
\mathbf{r}'(t) = \hat{i} - 3t^2 \hat{j} + 2t \hat{k}
\]

\[
\mathbf{v} = \|\mathbf{r}'(t)\| = \sqrt{1 + 9t^4 + 4t^2}
\]

\[
\mathbf{a} = \frac{d^2\mathbf{v}}{dt^2} = \frac{1}{2} \left( 1 + 9t^4 + 4t^2 \right)^{-\frac{1}{2}} (36t^3 + 8t)
\]

The point \((1, -1, 1)\) corresponds to \( t = 1 \).

So \( \mathbf{a}_T = \frac{1}{2} \cdot 14^{-\frac{1}{2}} \cdot 44 = \frac{22}{\sqrt{14}} \).

\[
\mathbf{a} = \mathbf{r}'' = 0 \hat{i} - 6t \hat{j} + 2 \hat{k}
\]

\[
\|\mathbf{a}\| = \sqrt{36t^2 + 4}.
\]

At \( t = 1 \) we get \( \mathbf{a}(1) = \sqrt{40} \)

\[
\mathbf{a}_N = \sqrt{\|\mathbf{a}\|^2 - \mathbf{a}_T^2} = \sqrt{40 - 22^2/14} = \sqrt{40 - 484/14} = \sqrt{35}
\]

\[
\mathbf{a}_N = \frac{2\sqrt{5}}{7} = \frac{\sqrt{35}}{7}
\]
3. (4 points) Write down the integral for the arc length of the curve \( \mathbf{r}(t) = \sin t \mathbf{i} - \cos 2t \mathbf{j} + t^2 \mathbf{k} \) between \( t = 2 \) and \( t = 4 \). Do not attempt to evaluate this integral.

\[
\mathbf{r}'(t) = \cos t \mathbf{i} + 2 \sin 2t \mathbf{j} + 2t \mathbf{k}
\]

\[
||\mathbf{r}'(t)|| = \sqrt{\cos^2 t + 4 \sin^2 2t + 4t^2}
\]

\[
L = \int_2^4 ||\mathbf{r}'(t)|| \, dt
\]

\[
= \int_2^4 \sqrt{\cos^2 t + 4 \sin^2 2t + 4t^2} \, dt.
\]
4. (6 points) Sketch the graph of the function $z = f(x, y) = x^2 + y^2$ on this three-dimensional set of axes. Be sure to exhibit your calculations of level sets, and be sure to label the level sets in your graph.

<table>
<thead>
<tr>
<th>( z )</th>
<th>level set</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x^2 + y^2 = 0 )</td>
<td>point</td>
</tr>
<tr>
<td>1</td>
<td>( x^2 + y^2 = 1 )</td>
<td>circle of r = 1</td>
</tr>
<tr>
<td>4</td>
<td>( x^2 + y^2 = 4 )</td>
<td>circle of r = 2</td>
</tr>
<tr>
<td>8</td>
<td>( x^2 + y^2 = 8 )</td>
<td>circle of r = 3</td>
</tr>
</tbody>
</table>
5. (6 points) Sketch the locus of points (the cylinder) described by the equation $y^2 + 4z^2 = 4$. 
6. (6 points) Let \( f(x, y) = x^2 \sin(xy) \). Calculate these partial derivatives:

(a) \( \frac{\partial f}{\partial y} = 2x \cdot y \cdot \cos(xy) \)

(b) \( \frac{\partial^2 f}{\partial x \partial y} = 3x^2 \cos(xy) \)
\( \frac{\partial^2 f}{\partial y \partial x} = 3x^2 \cos(xy) - x^3 y \sin(xy) \)

7. (8 points) The parametric equations for the tangent line to the curve \( r(t) = t^2 \mathbf{i} - t^3 \mathbf{j} + 3t \mathbf{k} \) at the point \((4, -8, 6)\) are given by (circle your answer)

(a) \[
\begin{align*}
x &= 3 + 3t \\
y &= -8 - 10t \\
z &= 6 + 2t 
\end{align*}
\]

(b) \[
\begin{align*}
x &= 4 + 4t \\
y &= -8 - 12t \\
z &= 6 + 3t 
\end{align*}
\]
(c)

\[ x = 4 + 3t \]
\[ y = -7 - 12t \]
\[ z = 6 + 2t \]

\[ \mathbf{r}'(t) = 2t \hat{i} + 3t^2 \hat{j} + 3 \hat{k} \]. The point corresponds to \( t = 2 \).

So \[ \mathbf{r}'(2) = 4 \hat{i} + 12 \hat{j} + 3 \hat{k} \]. The requested line is

\[ x = 4 + 4t \]
\[ y = -8 + 12t \]
\[ z = 6 + 3t \]

8. (8 points) The curvature of the curve \( \mathbf{r}(t) = t \hat{i} + t^2 \hat{j} - \cos t \hat{k} \) at the point \((0, 0, -1)\) is given by (circle your answer)

(a) \( \sqrt{2} \)
(b) \( \sqrt{4} \)
(c) \( \sqrt{5} \)
(d) \( \sqrt{3} \)

The curvature is

\[ \kappa \left( t \right) = \left| \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\left| \mathbf{r}'(t) \right|^3} \right| \]

\[ \left| \mathbf{r}'(0) \times \mathbf{r}''(0) \right| = \sqrt{4t^2 \cos^2 t + 4 \sin^2 t - 8t \cos t \sin t + \cos^2 t + 4} \]

\[ \left| \mathbf{r}'(0) \right| = \sqrt{1 + 4t^2 + \sin^2 t} \]. The point corresponds to \( t = 0 \).

\[ \left| \mathbf{r}'(0) \times \mathbf{r}''(0) \right| = \sqrt{1 + 0 + 0 + 0 + 4} = \sqrt{5} \]

\[ \kappa = \frac{\sqrt{5}}{1} = \sqrt{5} \]
9. (6 points) For the function $g(x, y) = xy^2 - yx^2$ and the direction vector $u = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$, the value of $D_u g(1, 1)$ is (circle your answer)

(a) $-\sqrt{3}$
(b) $\sqrt{2}$
(c) $-\sqrt{2}$
(d) $\sqrt{3}$

\[ D_u g(1, 1) = \nabla g(1, 1) \cdot u = \left( y^2-2xy, 2xy - x^2 \right) \bigg|_{(1,1)} \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \]

\[ = \langle -1, 1 \rangle \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \]

10. (8 points) For the function $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq 0 \\ 1 & \text{if } (x, y) = 0 \end{cases}$
the limit \( \lim_{(x,y) \to (0,0)} f(x, y) \) (circle your answer)

(a) exists
(b) does not exist
(c) is trying to exist
(d) may be forbidden to exist

\[
\begin{align*}
\lim_{t \to 0} f(t, 0) &= 0 \\
\lim_{t \to 0} f(t, t) &= \frac{1}{2} \\
\text{unequal so limit does not exist.}
\end{align*}
\]

11. (8 points) TRUE or FALSE: If \( f(x, y) = x^2 - y^3 \) and if \( x = s^2 - t^3 \) and \( y = s^3 + t^2 \), then

\[
\frac{\partial f}{\partial s} = 4s^3 - 4st^3 - 8s^8 - 10s^5t^2 - 9s^2t^4.
\]

Put your answer here:

\( F \)
12. (8 points) TRUE or FALSE: The tangent plane to the graph of \( z = f(x, y) = x^2 + y^2 \) at the point \((1, 2, 5)\) is given by

\[ 2x - 4y + 8z = 12. \]

Put your answer here:

\[ \boxed{F} \]

\[ \nabla f = \langle 2x, 2y \rangle \]

At the point \((1, 2, 5)\) we have \(\nabla f = \langle 2, 4 \rangle\).

The tangent plane is

\[ \langle 2, 4, -1 \rangle \cdot \langle x, y, z \rangle = 0 \]

where \( p = (1, 2, 5) \)

\[ \bar{X} = (x, y, t) . \]

This is

\[ \langle 2, 4, -1 \rangle \cdot \langle x - 1, y - 2, z - 5 \rangle = 0 \]

\[ 2x + 4y - z = 5 \]
13. (8 points) The gradient of the function \( f(x, y) = xy \cos x^2 \) is (circle your answer)

(a) \( (y \cos x^2 - 2y^2 \sin x^2, x \cos x^2) \)

(b) \( (x \cos y^2 - 2y^2 \sin y^2, y \cos y^2) \)

(c) \( (y \sin x^2 - 2x^2 \cos x^2, x \sin x^2) \)

(d) \( (x \cos x^2 - 2y^2 \sin x^2, y \sin x^2) \)

\[
\nabla f = \langle y \cos x^2 - xy, 2x \sin x^2, x \cos x^2 \rangle 
\]

14. (6 points) TRUE or FALSE: The domain of the function \( g(x, y) = \sqrt{4 - x^2 - y^4} \) is the set \( S = \{(x, y) : x^2 + y^4 \leq 4\} \). Put your answer here:

\[
\text{For } (x, y) \in S, \quad 4 - x^2 - y^4 \geq 0 \quad \text{so} \quad \sqrt{4 - x^2 - y^4} \text{ makes sense.}
\]